CZECH TECHNICAL UNIVERSITY IN PRAGUE

Mathematical modeling of laminar-turbulent transition with RANS approach

Doc. Ing. Jiří Fürst, PhD.

Dept. of Technical Mathematics, Faculty of Mechanical Engineering,

Czech Technical University in Prague,

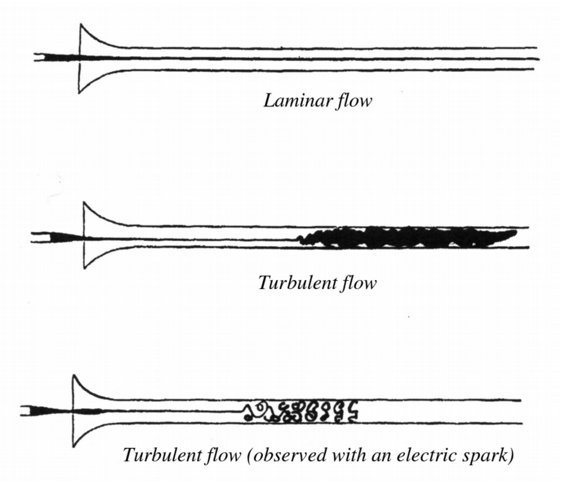
Technicka Street 4,

16607, Prague 6, Czech Republic.

Outline

- Description of laminar-turbulent transition
- Survey of mathematical models
  - Orr-Sommerfeld eq., e^n model
  - intermittency models (algebraic, transport eq.)
  - laminar kinetic energy models
- Pressure sensitive laminar kinetic energy model

Laminar-turbulent transition



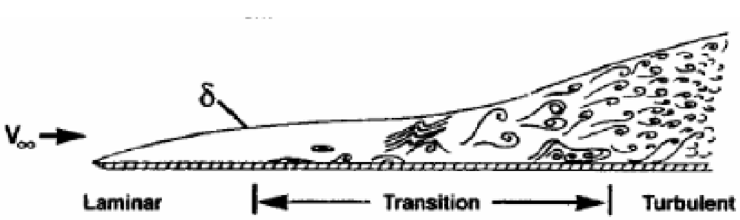
[O. Reynolds, 1883]

$$Re = \frac{UD}{\nu} < Re_{crit}$$

$$Re = \frac{UD}{\nu} > Re_{crit}$$

For pipes: $Re_{crit} \approx 2300$

Other experiments: $Re_{crit} \approx 2\,000 - 40\,000$

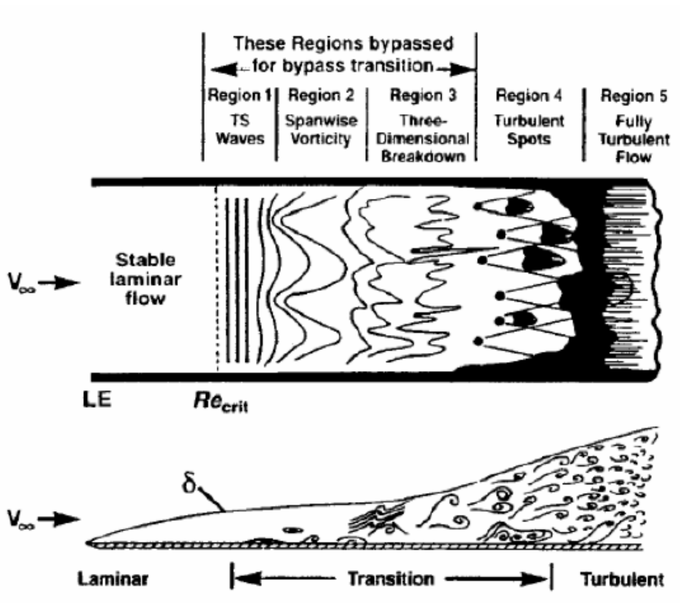


[Schlichting, 2000]

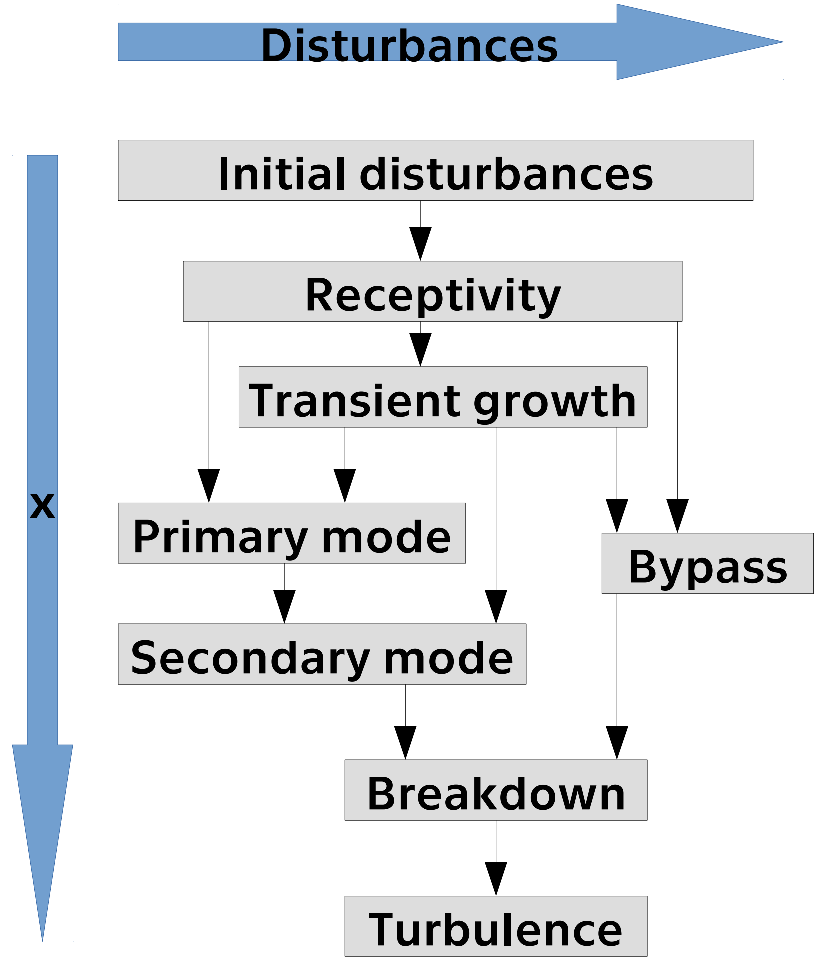
For flat plate:

$$Re_{crit} = \frac{Ux}{\nu} \approx 3.5 \times 10^5 - 10^6$$

Laminar-turbulent transition

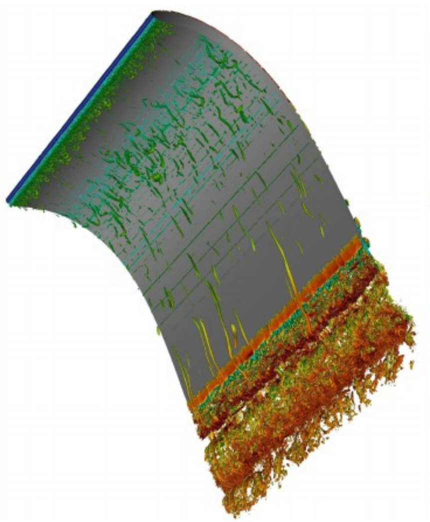


Natural transition according to Schlichting

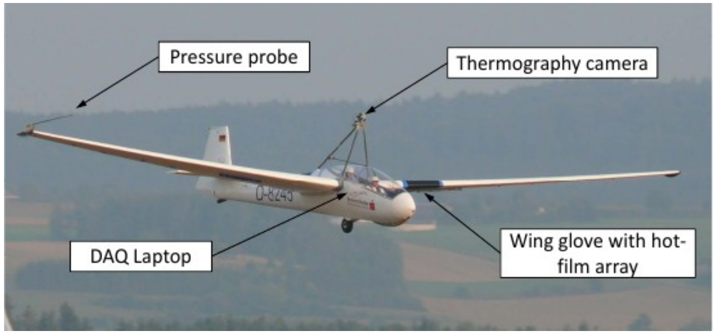


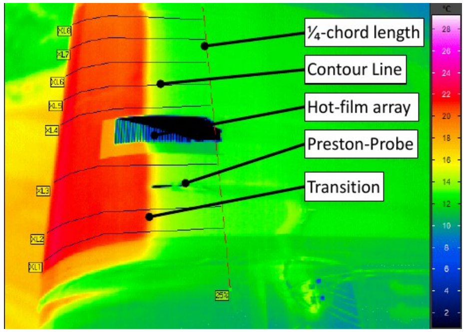
Path to transition according to Morkovin

Laminar-turbulent transition



DNS of flows over suction side of turbine blade, λ_2, [Hosseini et al, 2015]





Transition at the wings of a glider, infrared image [Schreivogel, 2015]

Mathematical models

Boundary layer codes:

- Stability of the B-L via Orr-Sommerfeld equation
- e^n method [van Ingen, 1956]

Solution of full Navier-Stokes equations:

Direct numerical simulation (DNS)

- Kolmogorov scale ~ *1/Re* => *O(Re³)* DoFs
- Very expensive!

Large eddy simulation (LES)

- Large eddies are simulated, small eddies are modeled
- For boundary layer transition => wall resolved LES => similar to DNS
- Hybrid RANS-LES (DES) simulation: RANS in the BL + LES in the farfield

Reynolds averaged Navier-Stokes (RANS)

- Usually calibrated for fully turbulent cases
- Needs some additional corrections / sub-models for transitional flows

Boundary layer eq. + e^N

Prandtl's equations:

$$uu_x + vu_y = -p_x + \nu u_{yy}$$
$$u_x + v_y = 0$$
$$p_y = 0$$

Boundary layer eq.:

$$\frac{d\theta}{dx} + (2+H)\frac{\theta}{U}\frac{dU}{dx} = \frac{c_f}{2},$$
$$\frac{d\theta^*}{dx} + 3\frac{\theta^*}{U}\frac{dU}{dx} = c_D$$

Boundary layer parameters:

$$\delta := \int_0^\infty \left(1 - \frac{u}{U}\right) dy$$
$$\theta := \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$$
$$\theta^* := \int_0^\infty \frac{u}{U}\left(1 - \left(\frac{u}{U}\right)^2\right) dy$$
$$H := \frac{\delta}{\theta}$$

Closure: $c_f = c_f(H,\theta),\ c_D = c_D(H,\theta)$

Laminar BL ($x<x_{tr}$):

- **from Blasius or Falkner-Skann solutions**

Turbulent BL ($x>x_{tr}$):

- **from power law**
- **from correlations**

Value of x_{tr}?

Orr-Sommerfeld equation

Assume:

$$u(x,y,t) = U(y) + u'(x,y,t),$$
$$v(x,y,t) = 0 + v'(x,y,t),$$
$$p(x,y,t) = P(x) + p'(x,y,t).$$

Note: $\frac{1}{\rho}P_x = \nu U_{yy}$

Linearized NS:

$$u'_t + Uu'_x + v'U_y + \frac{1}{\rho}p'_x = \nu\nabla^2 u',$$
$$v'_t + Uv'_x + \frac{1}{\rho}p'_y = \nu\nabla^2 v',$$
$$u'_x + v'_y = 0.$$

Introducing stream function: $u' = \Psi_y,\ v' = -\Psi_x$

$$\Psi_{yt} + U\Psi_{yx} - U_y\Psi_x + \frac{1}{\rho}p_x = \nu(\Psi_{yxx} + \Psi_{yyy}),$$
$$-\Psi_{xt} - U\Psi_{xx} + \frac{1}{\rho}p_y = \nu(-\Psi_{xxx} - \Psi_{xyy}).$$

$$\Downarrow$$

$$\nabla^2\Psi_t + U(\Psi_{xxx} + \Psi_{xyy}) - U_{yy}\Psi_x = \nu(2\Psi_{xxyy} + \Psi_{yyyy} + \Psi_{xxxx})$$

Orr-Sommerfeld equation

$$\Psi(x,y,t) = \phi(y)e^{i(\alpha x - \beta t)}$$

$$(U - \frac{\beta}{\alpha})(\phi'' - \alpha^2\phi) = -\frac{i\nu}{\alpha}(\phi'''' - 2\alpha^2\phi'' + \alpha^4\phi).$$

$$y = 0 \Rightarrow \phi = \phi' = 0,$$
$$y \to \infty \Rightarrow \phi = \phi' = 0.$$

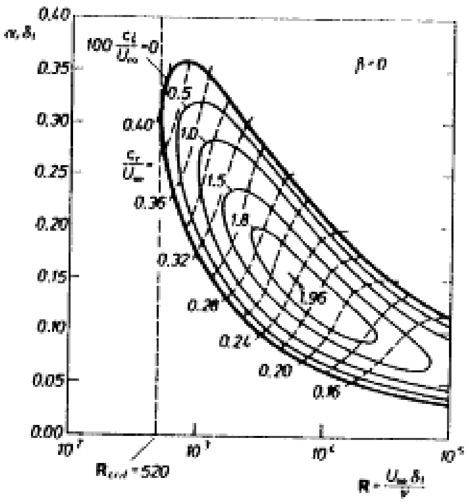
Temporal stability: $\alpha \in \mathbb{R},\ \beta = \beta_r + i\beta_i$

Spatial stability: $\alpha = \alpha_r + i\alpha_i,\ \beta \in \mathbb{R}$

Eigenvalue problem:
- **given *U, β, v***
- **find *α, φ***

If $\alpha_i < 0$, flow is unstable

O-S equation allows to find Re_{crit} and growth rate for given disturbance.



[Schlichting, 2000]

e^N model

Initial perturbation: $a_0 = ?$

Growth factor: $n(x,\beta) = \ln(a(x,\beta)/a_0) = \int_{x_0}^{x} -\alpha_i(\beta)\, d\xi.$

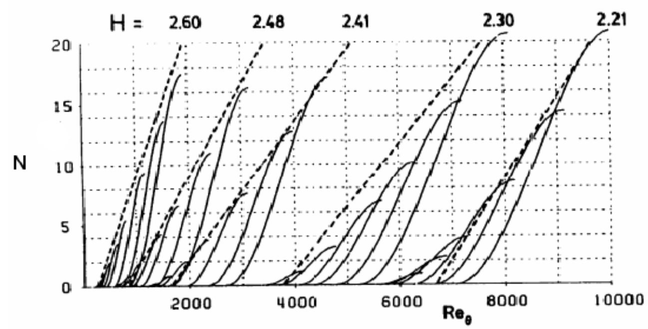
Transition location: $\max_{\beta} n(x_{tr},\beta) = n_{crit} \approx 9.$

van Ingen (*Tu>0.1%*):
- **transition start:** $n_1 = 2.13 - 6.18 \log(Tu)$
- **transition complete:** $n_2 = 5 - 6.18 \log(Tu)$

Envelope method [Drela]:

$$N = \frac{dN}{dRe_\theta}(Re_\theta - Re_{\theta,crit}),$$

$$\frac{dN}{dRe_\theta} = f(H),$$



$$log_{10}(Re_{\theta,crit}) = 2.492\left(\frac{1}{H-1}\right)^{0.43} + 0.7\left(\tanh(14\frac{1}{H-1} - 9.24) + 1\right)$$

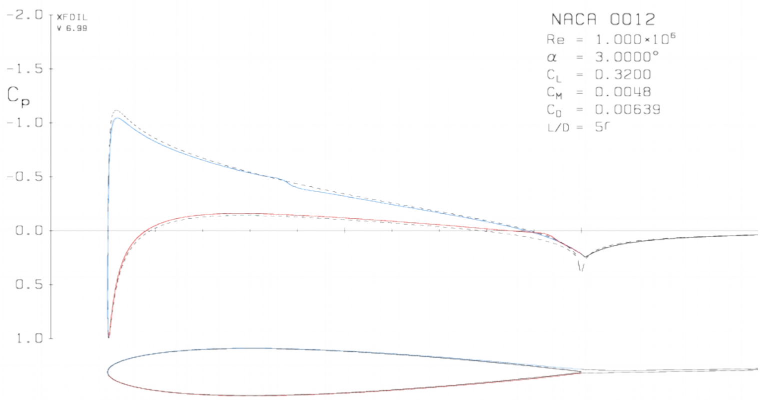
e^N model

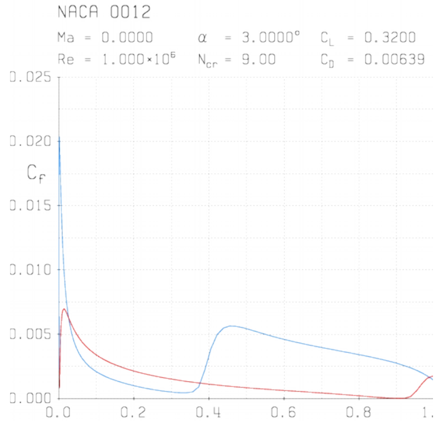
Pros:

- Simple and reasonably accurate model for natural transition
- Built on mathematical background (Orr-Sommerfeld eq.)
- Easy to couple with integral boundary layer methods and inviscid models (see e.g. XFoil package)
- Very useful for (2D) flows with low Tu (wind-turbines, sailplanes, ...)

Cons:

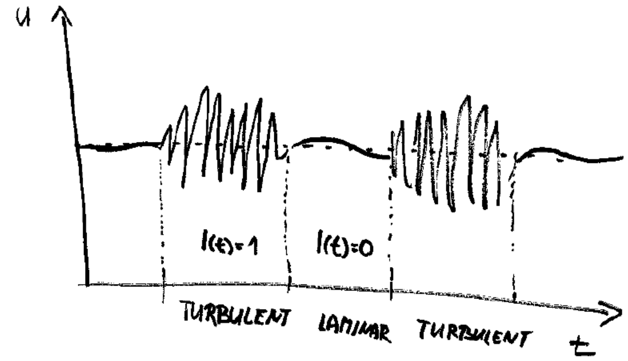
- Does not cover bypass transition
- Difficult do extend the model to complex 3D flows
- Not directly compatible with general N-S codes





Intermittency based models

Intermittency: $\gamma(x) := \lim_{T\to\infty} \frac{1}{T} \int_0^T I(t)\,dt$



For transition in boundary layer [Narasimha, Emmons]:

$$\gamma(x) = 1 - \exp\left(\frac{N\sigma}{U}(x - x_t)^2\right) = 1 - \exp\left(\hat{N}\sigma(Re_x - Re_{xt})^2\right).$$

Correlations for *N*, *σ*, eg. [Mayle], [Gostelow] $\hat{N}\sigma = 15 \cdot 10^{-12} Tu^{7/4} F(K)$

Correlations for x_t (Re_{xt}), eg. [Mayle], [Abu-Ghannam, Shaw], ...

$$Re_{\theta t} = 400 Tu^{-0.625}, \qquad \text{for } Tu > 1\%$$

$$Re_{\theta t} = 163 + \exp\left(F(\lambda_\theta) - \frac{F(\lambda_\theta)}{6.91} Tu\right), \qquad \text{for low } \ Tu$$

$$Re_\theta = \frac{U\theta}{\nu},\ \lambda_\theta = \frac{\theta^2}{\nu}\frac{dU}{ds}.$$

Algebraic intermittency model

Model by Příhoda & Straka:

- **Calculation of Re_θ via $Re_{v,max}$ assuming non-zero pressure gradient**
- **Improved correlations**

Coupling to a two-equation *k*-*ω* model:

$$\frac{Dk}{Dt} = \tilde{P}_k - \tilde{D}_k + \nabla(\nu_k^{eff}\nabla k),$$

$$\frac{D\omega}{Dt} = P_\omega - D_\omega - C_D + \nabla(\nu_\omega^{eff}\nabla\omega),$$

$$\tilde{P}_k = \gamma P_k,$$

$$\tilde{D}_k = \max(\gamma, 0.1) D_k,$$

$$\nu^{eff} = \nu + \gamma\nu_t.$$

$$\gamma = \frac{\gamma_i + \gamma_e}{2} + \frac{\gamma_e - \gamma_i}{2}\tanh\left[C_\gamma\left(\frac{d}{\delta_{995}} - 1\right)\right],$$

$$\gamma_i = \begin{cases} 1 - \exp\left[-\hat{n}\sigma(Re_x - Re_{xt})^2\right], & \text{for } Re_x > Re_{xt} \\ 0, & \text{otherwise.} \end{cases}$$

$$Re_{\theta t} = Re_{\theta t_0}\left[1 + F(Tu)\frac{1 - \exp(-40\lambda_t)}{1 + 0.4\exp(-40\lambda_t)}\right],$$

$$Re_{\theta t_0} = \begin{cases} 975.8 - 497.2Tu + \frac{11.4}{Tu} & \text{for } Tu \leq 1\,\%, \\ 96.7 + \frac{340}{Tu} + \frac{53.3}{Tu^2} & \text{for } Tu > 1\,\%. \end{cases}$$

$$F(Tu) = 0.29[1 - 0.54\exp(-3.5Tu)]\exp(-Tu),$$

$$N = \begin{cases} 0.86 \times 10^{-3} Tu^{-0.564}\exp\left(2.134\lambda_t \ln Tu - 59.23\lambda_t\right)], & \lambda_t < 0, \\ 0.86 \times 10^{-3} Tu^{-0.564}\exp\left(-10\sqrt{\lambda_t}\right), & \lambda_t \geq 0. \end{cases}$$

$$N = \hat{n}\sigma\, Re_{\theta t}^3$$

$$Re_{\nu\,\max} = \max(d^2|\Omega|/\nu),$$

$$K = \frac{\nu}{U_e^2}\frac{dUe(s)}{ds},$$

$$L = Re_{\nu\,\max}^2 K,$$

$$C = 2.185 - 5.79L + 63.076\min(0, L)^4,$$

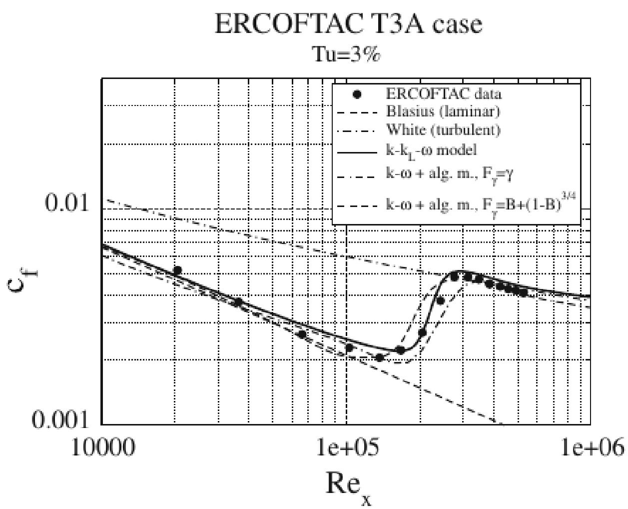
$$Re_\theta = \frac{Re_{\nu\,\max}}{C},$$

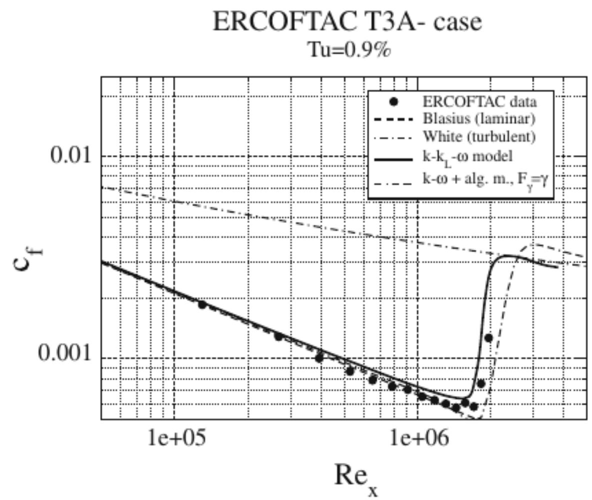
$$\lambda = Re_\theta^2 K,$$

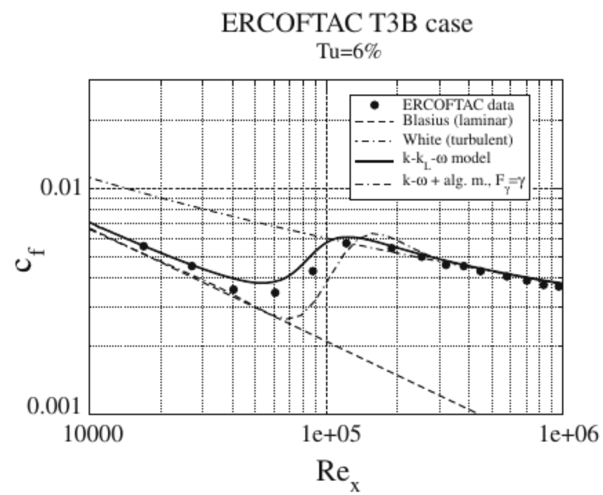
CTU CZECH TECHNICAL UNIVERSITY IN PRAGUE

Algebraic intermittency model

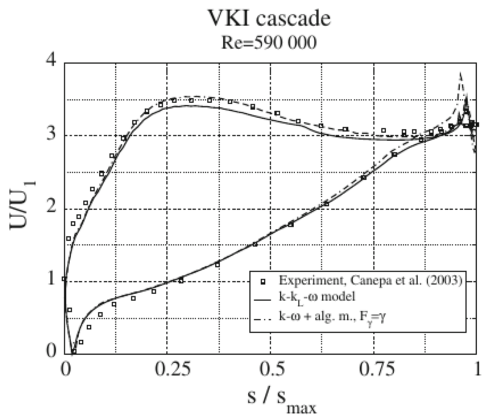
Flat plate flows [Fürst, Příhoda, Straka: Computing, 2013]

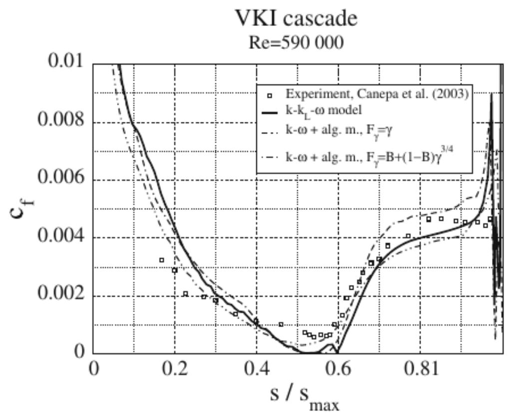


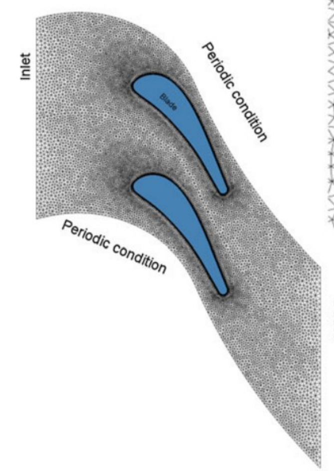


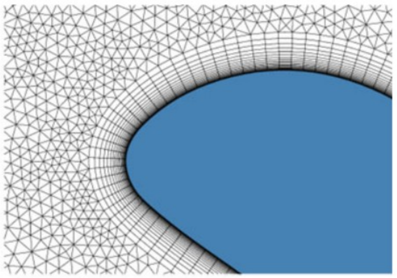


Flows through a turbine cascade [FPS, 2013]









Algebraic intermittency model

Pros:

- Able to cover natural and bypass transition
- Easy to test/implement custom correlations
- Extensible for other scenarios of the laminar-turbulent transition (transition in separated flows, ...)
- Computationally inexpensive (for simple geometry & structured meshes)

Cons:

- Very difficult implementation in the case of unstructured meshes
- Basically 2D model
- Does not reflect the history of the flow

Intermittency transport model

Concept of the model:

$$\gamma(x) = 1 - \exp\left(-\beta_\gamma^2 (x - x_t)^2\right) \quad \Rightarrow \quad \sqrt{-\ln(1-\gamma)} = \beta_\gamma (x - x_t)$$

$$\frac{d\gamma}{dx} = 2\beta_\gamma (1-\gamma)\sqrt{-\ln(1-\gamma)}$$

$$\vec{u}\cdot\nabla\gamma = 2\beta_\gamma ||\vec{u}||(1-\gamma)\sqrt{-\ln(1-\gamma)}$$

$$\frac{D\gamma}{Dt} = 2\sqrt{\hat{N}\sigma}\frac{U||u||}{\nu}(1-\gamma)\sqrt{-\ln(1-\gamma)}F_{onset} + \nabla(\nu_\gamma^{eff}\nabla\gamma)$$

$$\frac{D\gamma}{Dt} = F_{length} S (1-\gamma)\sqrt{\gamma} F_{onset} + \nabla(\nu_\gamma^{eff}\nabla\gamma), \text{ where } S = \sqrt{2S_{ij}S_{ij}}$$

Intermittency transport model

Langtry and Menter, 2006 and 2009:

$$\frac{D\gamma}{Dt} = P_\gamma - E_\gamma + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_\gamma}\right)\frac{\partial \gamma}{\partial x_j}\right]$$

$$P_\gamma = c_{a1} F_{length} S (1 - c_{e1}\gamma)\sqrt{\gamma F_{onset}},$$
$$E_\gamma = c_{a2} F_{turb} \Omega (c_{e2}\gamma - 1)\gamma$$

$$F_{turb} = F_{turb}(\nu_t/\nu)$$

From Blasius profile:

$$Re_\theta = \frac{\max_y Re_\nu}{2.193},\ Re_\nu = \frac{y^2 S}{\nu}$$

$$F_{onset} = F_{onset}\left(\frac{Re_\nu}{2.193 Re_{\theta c}}, \nu_t/\nu\right)$$

Correlation for F_{length} and $Re_{\theta c}$, [LM 2009]:

$$F_{length} = F_{length}(\overline{Re}_{\theta t}),$$
$$Re_{\theta c} = Re_{\theta c}(\overline{Re}_{\theta t}).$$

$$\frac{D\overline{Re}_{\theta t}}{Dt} = P_{\theta t} + \frac{\partial}{\partial x_j}\left[\sigma_{\theta t}(\nu + \nu_t)\frac{\partial \overline{Re}_{\theta t}}{\partial x_j})\right]$$

Intermittency transport model

Langtry and Menter model coupled with SST model:

Pros:

- Implemented in many software packages
- Possible to change/improve correlations for particular cases (low Tu, surface roughness, ...)
- Coupled to well established SST turbulence model

Cons:

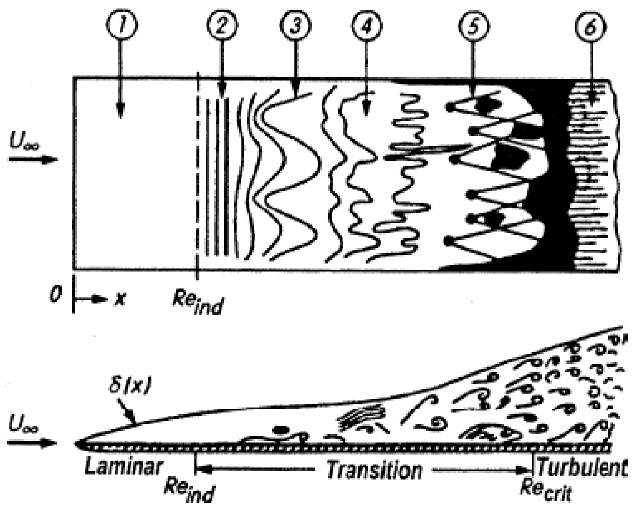
- Based on non-physical quantity $\overline{Re}_{\theta t}$
- Valid for Tu>0.027%, but needs specific tuning of inlet value of Tu and ω in order to capture some basic test cases

Extensions:

- Model for surface roughness induced transition [Dassler,2010], [Langel et al., 2014]
- Three-equation γ-SST model of Menter et al., 2015

Laminar kinetic energy model

- Concept by Mayle and Shultz
- Three-equation model based on k-ε [Walters, Leylek]
- Three-equation model based on k-ω [Walters, Cokljat]



(1) Laminar flow
(2) Tollmien-Schlichting waves (k_L)
(3) 3D vortices
(4) Vortex breakdown
(5) Turbulent spots (k_T)
(6) Fully turbulent flow

Laminar kinetic energy model

$$
\begin{aligned}
\frac{Dk_T}{Dt} &= P_{k_T} + R_{BP} + R_{NAT} - \omega k_T - D_T + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\alpha_T}{\sigma_k}\right)\frac{\partial k_T}{\partial x_j}\right], \\
\frac{Dk_L}{Dt} &= P_{k_L} - R_{BP} - R_{NAT} - D_L + \frac{\partial}{\partial x_j}\left[\nu \frac{\partial k_L}{\partial x_j}\right], \\
\frac{D\omega}{Dt} &= C_{\omega 1}\frac{\omega}{k_T}P_{k_T} + \left(\frac{C_{\omega R}}{f_W} - 1\right)\frac{\omega}{k_T}(R_{BP} + R_{NAT}) - C_{\omega 2}\omega^2 \\
&\quad + C_{\omega 3} f_\omega \alpha_T f_W^2 \frac{\sqrt{k_T}}{d^3} + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\alpha_T}{\sigma_\omega}\right)\frac{\partial \omega}{\partial x_j}\right].
\end{aligned}
$$

Two length scales of turbulence:

- $\lambda_{eff} = \min(C_\lambda d, \lambda_T),\ \lambda_T = \sqrt{k_T}/\omega,\ k_{T,s} = f_{SS}\left(\lambda_{eff}/\lambda_T\right)^{2/3}.$
- Small vortices => P_{kT}, large vortices => P_{kL}

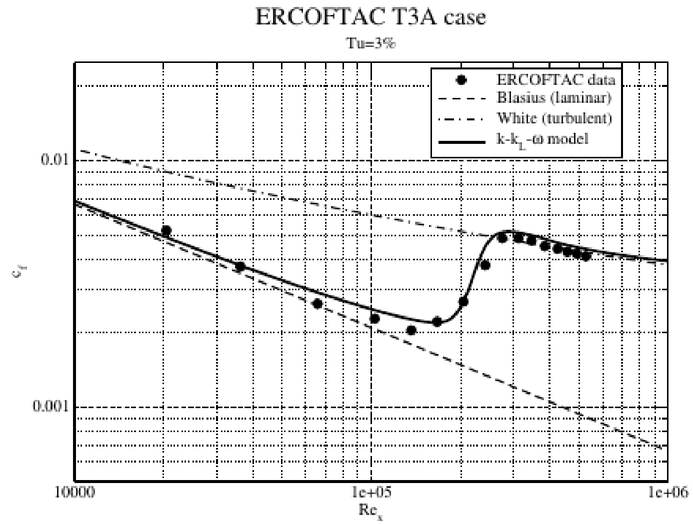
Energy transfer from k_L to k_T:

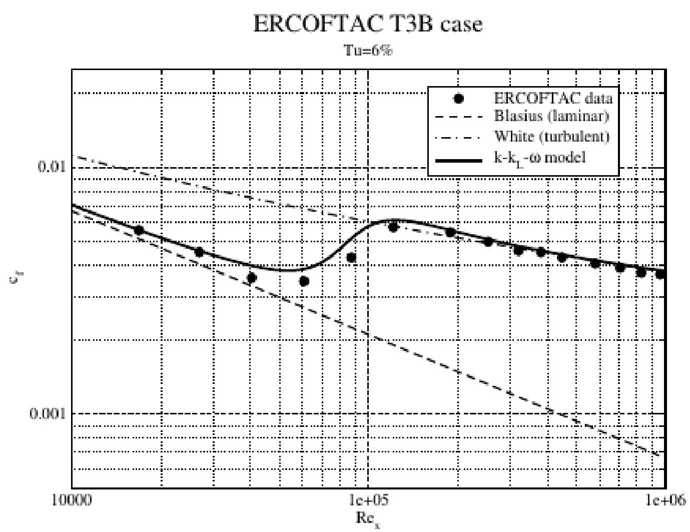
- Natural transition: R_{NAT} active when $Re_\Omega = d^2\Omega/v > C_{NATcrit}$
- Bypass transition: R_{BP} active when $k_T/(v\Omega) > C_{BPcrit}$

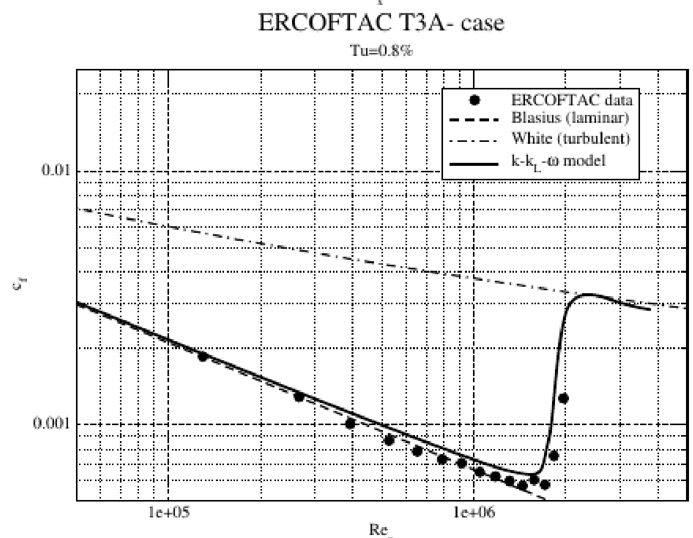
Laminar kinetic energy model

ERCOFTAC test cases for flat plate flows

- T3A $Tu = 3\%$
- T3B $Tu = 6\%$
- T3A- $Tu = 0.8\%$







CTU
CZECH TECHNICAL UNIVERSITY IN PRAGUE

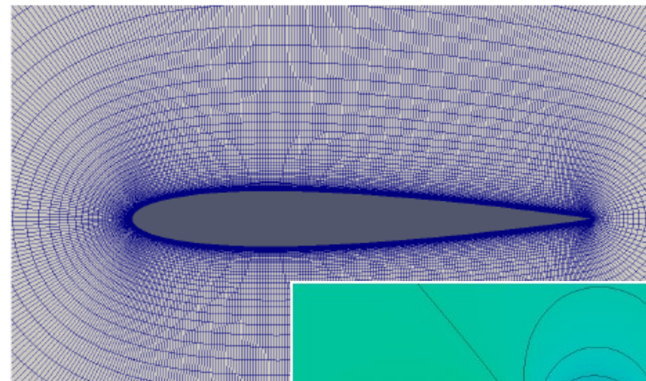
Laminar kinetic energy model

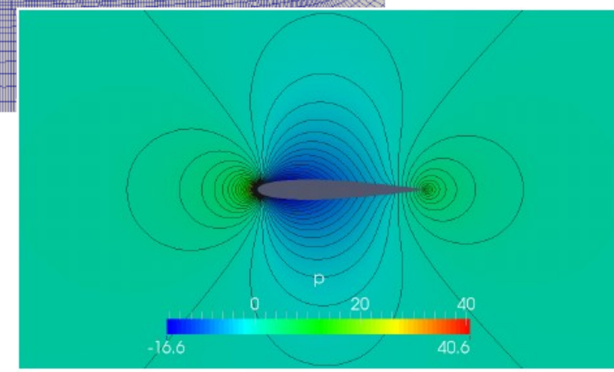
Flows over NACA 0012 profile

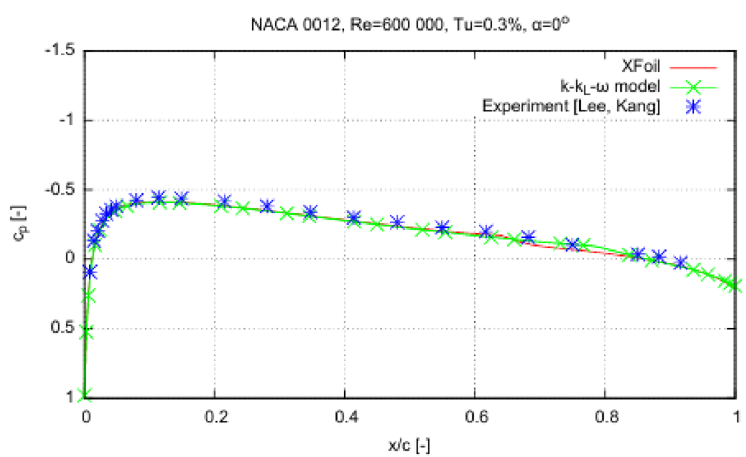
- *Re=600 000, Tu=0.3%, AoA=0º*
- Fine mesh with 500x100 cells
- 2nd order FVM method (OpenFOAM)

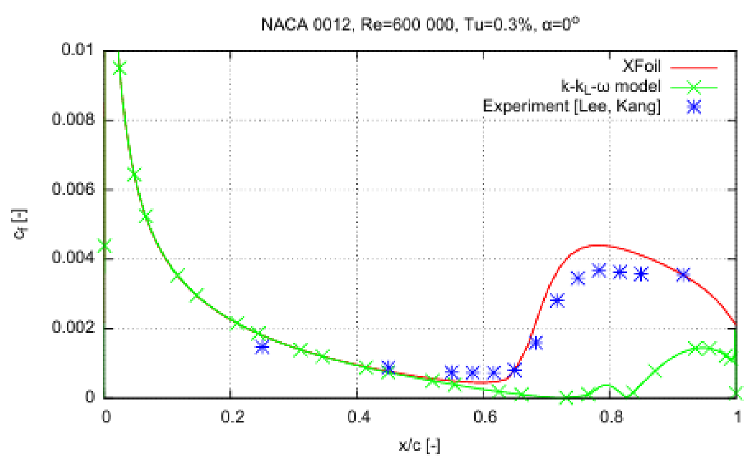
Comparison with

- Experiment [Lee, Kang]
- XFoil (e^n) [Drella]



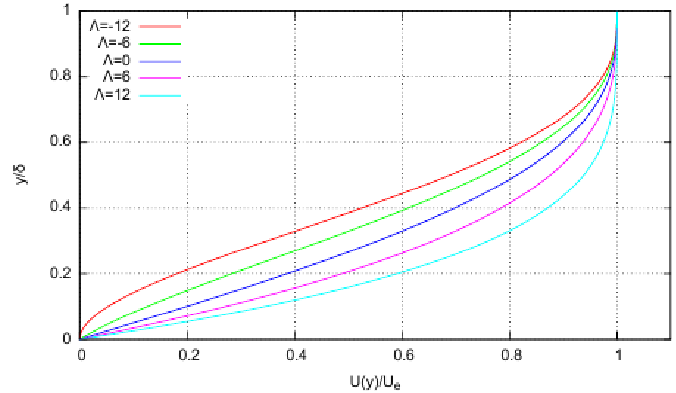






LKE model for APG flows

- The original Walters and Cokljat model needs a modification for flows with adverse pressure gradient (APG) at low *Tu.*
- New thresholds for R_{NAT} and P_{kL} [Fürst et al., 2015]



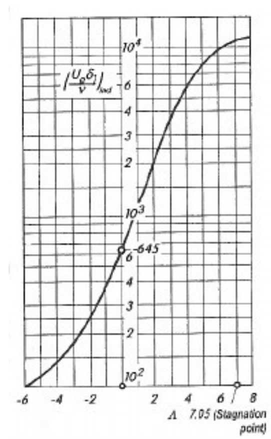
Pohlhausen velocity profile:

$$\frac{U(y)}{U_e} = 2\eta - 2\eta^3 + \eta^4 + \frac{\Lambda}{6}\eta(1-\eta)^3$$

$$\eta = y/\delta,\ \Lambda = \frac{\delta^2}{\nu}\frac{dU_e}{dx}.$$

Stability limit for Pohlhausen profiles:

- Orr-Sommerfeld eq., [Schlichting, Ulrich, 1942]
- Stability limit $Re_{ind}=f(\Lambda)$

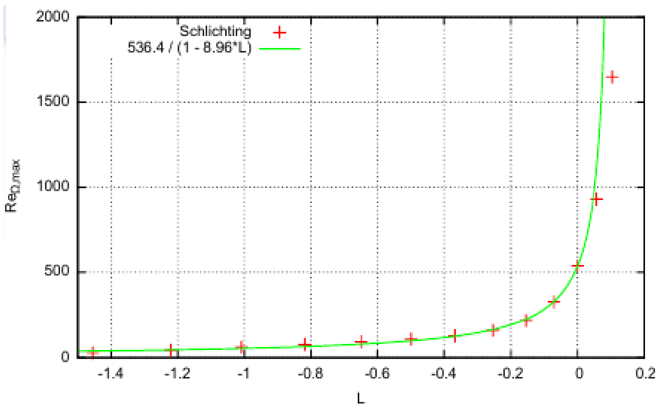


LKE model for APG flows

- **Difficult calculation of δ and Λ => reformulated in terms of Re_Ω and L**

$$Re_{\Omega,ind} \approx \frac{536.4}{1-8.963L}, \text{ for } -1.5 \leq L \leq 0.$$

$$L = Re^2_{\Omega,max} \frac{\nu}{U_e^2} \frac{dU_e}{dx}$$

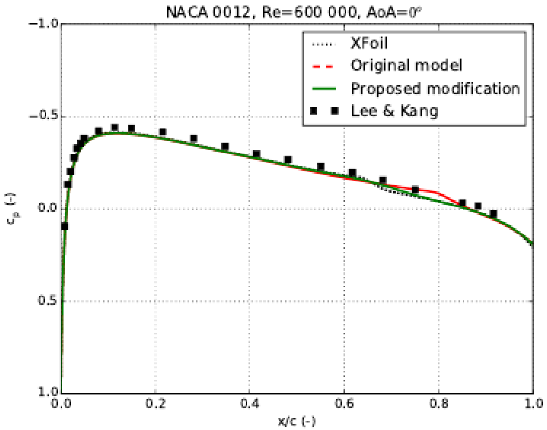


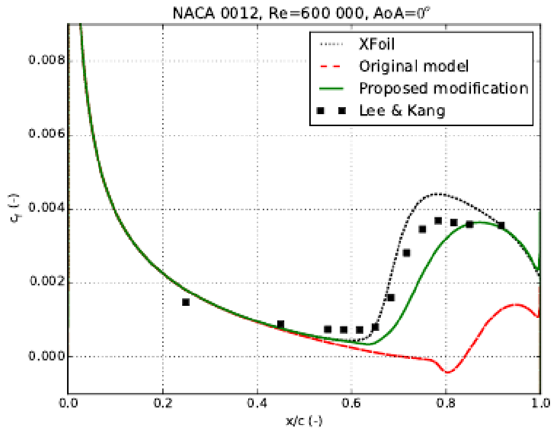
New thresholds for APG flows:

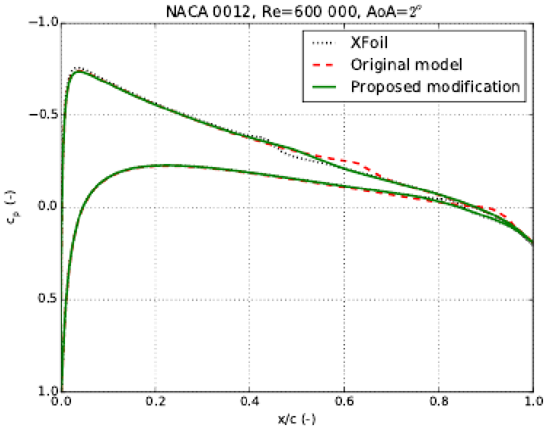
- **Threshold for P_{kL}:** $$C^{APG}_{TS,crit} = Re_{\Omega,ind} = \frac{536.4}{1-8.963L}$$
- **Threshold for R_{NAT}:** $$C_{NAT,crit} = \frac{1250}{1-8.963L}.$$

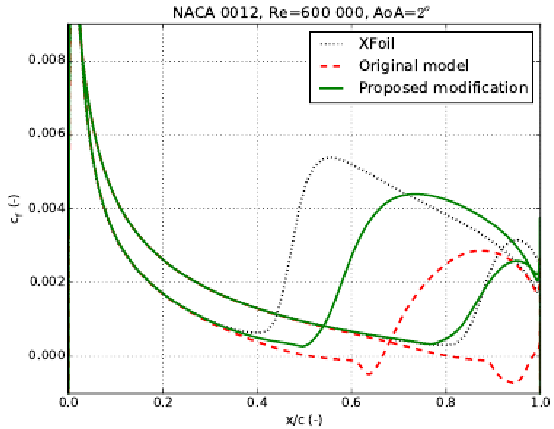
LKE model for APG flows

Flows over NACA 0012 profile, Re=600 000, Tu=0.3%



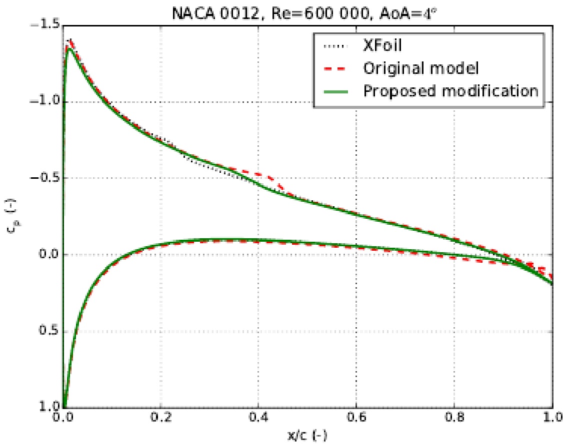


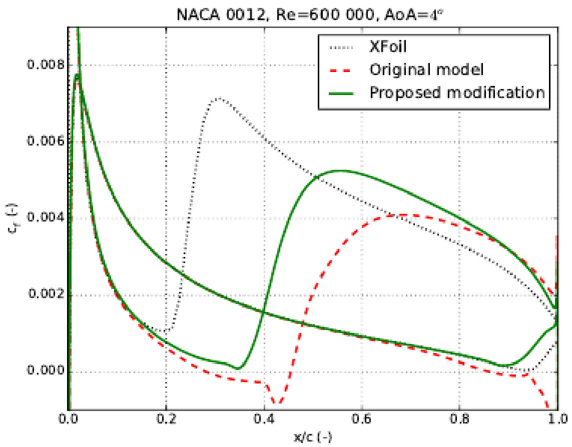


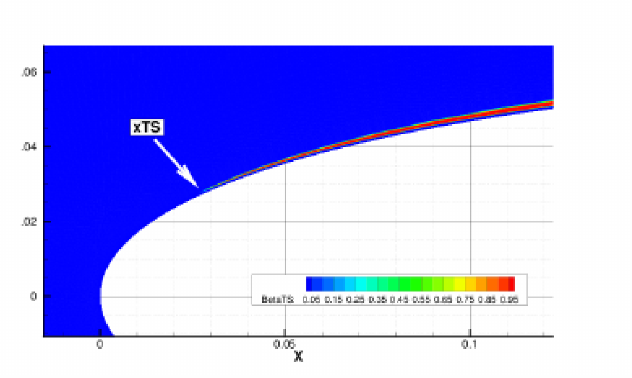


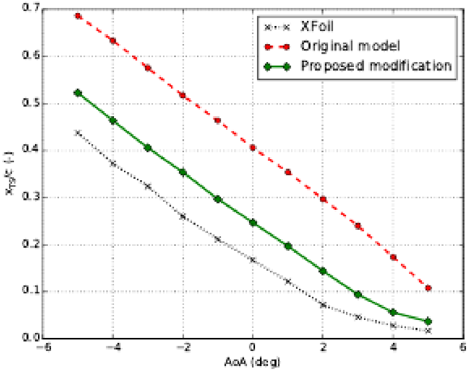
LKE model for APG flows

Flows over NACA 0012 profile, Re=600 000, Tu=0.3%





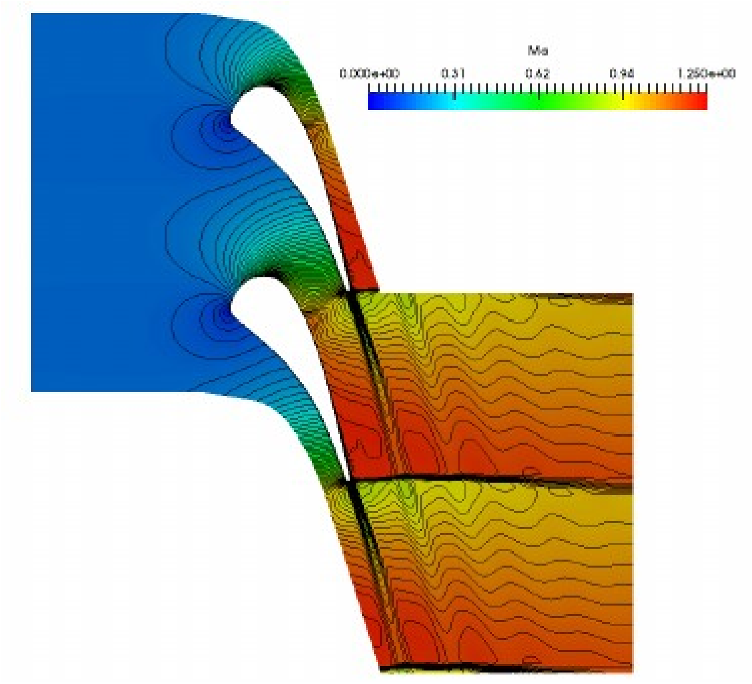


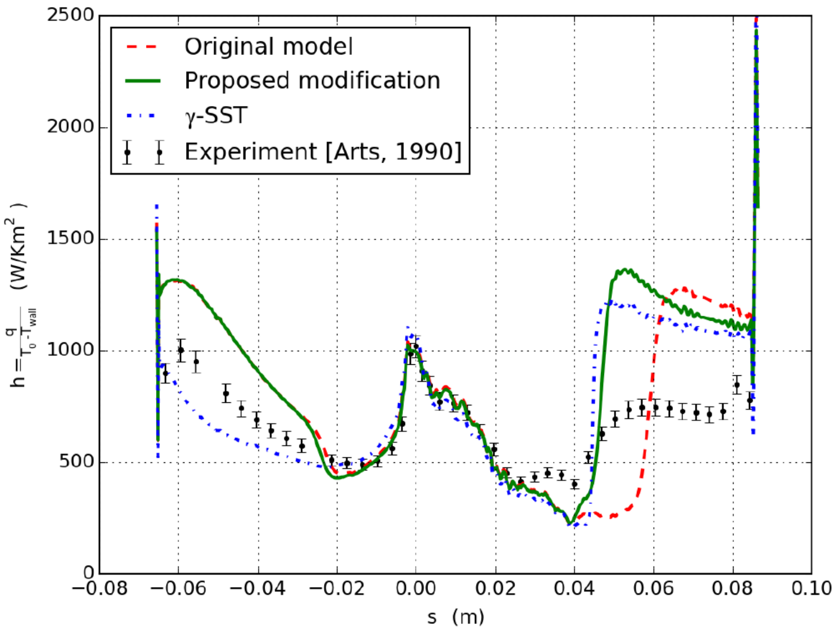


LKE model for APG flows

Flows through the VKI cascade, Re=2 110 000, M_{2i} = 1.089 (case 241)

- Experimental data by Arts
- Fixed wall temperature, measurements of heat flux

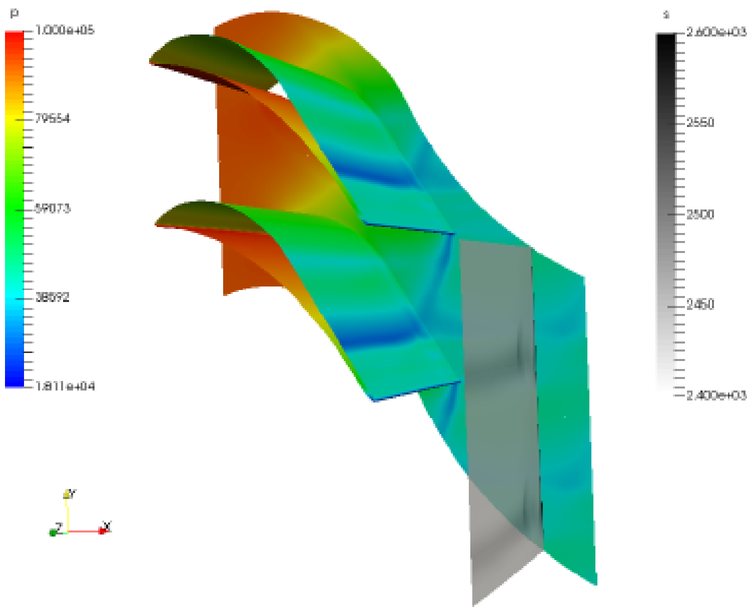


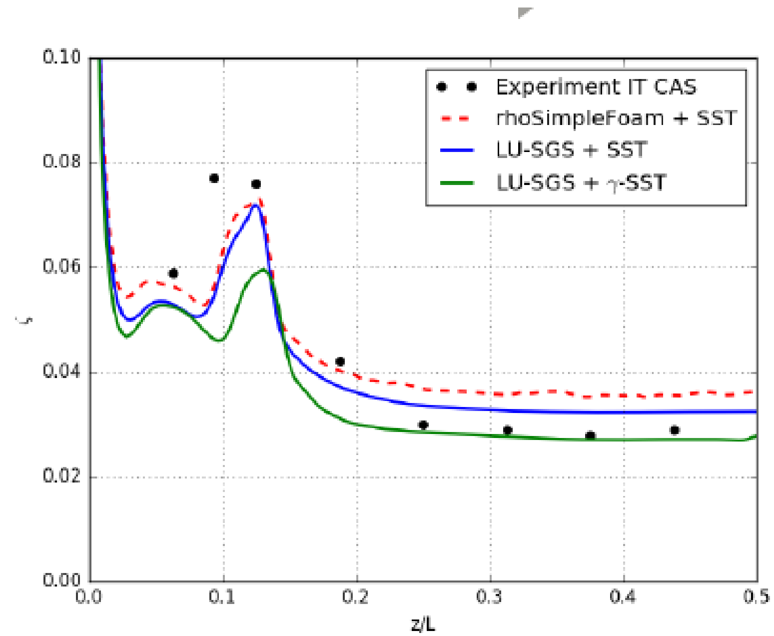


Transitional flow in 3D case

Flows through 3D prismatic cascade (TR-L-1), Re=1 200 000, M_{2i}=1.2

- Experimental data by IT CAS





Span-wise distribution of kinetic energy loss

Conclusions

- Laminar-turbulent transition plays an important role in accurate prediction of flows in many engineering applications
- There exist a lot of ways from laminar to turbulent boundary layer (natural transition, bypass transition, transition in LSB, ...)
- There is no model capturing all transition scenarios!
- Existing RANS based transition models heavily rely on the experimental data via correlations.

Future of RANS based transition models

- DNS – still too expensive
- LES – for proper transition modeling on needs to resolve small scale fluctuations in the vicinity of the wall => almost as expensive as DNS
- Hybrid RANS-LES (DES) – combination of RANS model in the vicinity of the wall with LES in freestream => necessity of RANS based model.

Acknowledgement: This lecture was supported by the Grant Agency of the Czech Technical University in Prague, grant No. SGS 13/174/OHK2/3T/12