Mathematical modeling of laminar-turbulent transition with RANS approach

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Outline

• Description of laminar-turbulent transition

• Survey of mathematical models
  • Orr-Sommerfeld eq., $e^n$ model
  • intermittency models (algebraic, transport eq.)
  • laminar kinetic energy models

• Pressure sensitive laminar kinetic energy model
Laminar-turbulent transition

\[ Re = \frac{UD}{\nu} < Re_{crit} \]

\[ Re = \frac{UD}{\nu} > Re_{crit} \]

For pipes: \( Re_{crit} \approx 2300 \)

Other experiments: \( Re_{crit} \approx 2000 - 40000 \)

For flat plate:

\[ Re_{crit} = \frac{Ux}{\nu} \approx 3.5 \times 10^5 - 10^6 \]

[O. Reynolds, 1883]

[Schlichting, 2000]
Laminar-turbulent transition

Initial disturbances
- Receptivity
  - Transient growth
    - Primary mode
    - Secondary mode
      - Breakdown
        - Turbulence

Natural transition according to Schlichting

Path to transition according to Morkovin
Laminar-turbulent transition

Transition at the wings of a glider, infrared image [Schreivogel, 2015]

DNS of flows over suction side of turbine blade, $\lambda_2$, [Hosseini et al, 2015]
Mathematical models

**Boundary layer codes:**
- Stability of the B-L via Orr-Sommerfeld equation
- $e^n$ method [van Ingen, 1956]

**Solution of full Navier-Stokes equations:**

Direct numerical simulation (DNS)
- Kolmogorov scale $\sim 1/Re \Rightarrow O(Re^3)$ DoFs
- Very expensive!

Large eddy simulation (LES)
- Large eddies are simulated, small eddies are modeled
- For boundary layer transition $\Rightarrow$ wall resolved LES $\Rightarrow$ similar to DNS
- Hybrid RANS-LES (DES) simulation: RANS in the BL + LES in the farfield

Reynolds averaged Navier-Stokes (RANS)
- Usually calibrated for fully turbulent cases
- Needs some additional corrections / sub-models for transitional flows
Prandtl's equations:

\[ uu_x + vu_y = -p_x + \nu u_{yy} \]
\[ u_x + v_y = 0 \]
\[ p_y = 0 \]

Boundary layer parameters:

\[ \delta := \int_0^\infty \left( 1 - \frac{u}{U} \right) dy \]
\[ \theta := \int_0^\infty \frac{u}{U} \left( 1 - \frac{u}{U} \right) dy \]
\[ \theta^* := \int_0^\infty \frac{u}{U} \left( 1 - \left( \frac{u}{U} \right)^2 \right) dy \]
\[ H := \frac{\delta}{\theta} \]

Boundary layer eq.:

\[ \frac{d\theta}{dx} + (2 + H) \frac{\theta}{U} \frac{dU}{dx} = \frac{c_f}{2} \]
\[ \frac{d\theta^*}{dx} + 3 \frac{\theta^*}{U} \frac{dU}{dx} = c_D \]

Closure: \[ c_f = c_f(H, \theta), \ c_D = c_D(H, \theta) \]

Laminar BL (x<x\textsubscript{tr}):
- from Blasius or Falkner-Skan solutions

Turbulent BL (x>x\textsubscript{tr}):
- from power law
- from correlations

Value of x\textsubscript{tr}?
Orr-Sommerfeld equation

Assume:
\[ u(x, y, t) = U(y) + u'(x, y, t), \]
\[ v(x, y, t) = 0 + v'(x, y, t), \]
\[ p(x, y, t) = P(x) + p'(x, y, t). \]

Note:
\[ \frac{1}{\rho} P_x = \nu U_{yy} \]

Linearized NS:
\[ u_t' + U u_x' + v' U_y + \frac{1}{\rho} p_x' = \nu \nabla^2 u', \]
\[ v_t' + U v_x' + \frac{1}{\rho} p_y' = \nu \nabla^2 v', \]
\[ u_x' + v_y' = 0. \]

Introducing stream function:
\[ u' = \Psi_y, \quad v' = -\Psi_x \]
\[ \Psi_{yt} + U \Psi_{yx} - U_y \Psi_x + \frac{1}{\rho} p_x = \nu (\Psi_{yxx} + \Psi_{yyy}), \]
\[ -\Psi_{xt} - U \Psi_{xx} + \frac{1}{\rho} p_y = \nu (-\Psi_{xxx} - \Psi_{xyy}). \]

\[ \nabla^2 \Psi_t + U (\Psi_{xxx} + \Psi_{xyy}) - U_{yy} \Psi_x = \nu (2 \Psi_{xyyy} + \Psi_{yyyy} + \Psi_{xxxx}) \]
Orr-Sommerfeld equation

\[ \Psi(x, y, t) = \phi(y)e^{i(\alpha x - \beta t)} \]

\[ (U - \frac{\beta}{\alpha})(\phi'' - \alpha^2 \phi) = -\frac{i\nu}{\alpha}(\phi'''' - 2\alpha^2 \phi'' + \alpha^4 \phi). \]

Temporal stability: \( \alpha \in \mathbb{R}, \beta = \beta_r + i\beta_i \)
Spatial stability: \( \alpha = \alpha_r + i\alpha_i, \beta \in \mathbb{R} \)

Eigenvalue problem:
- given \( U, \beta, \nu \)
- find \( \alpha, \phi \)

If \( \alpha_i < 0 \), flow is unstable

O-S equation allows to find \( \text{Re}_{\text{crit}} \) and growth rate for given disturbance.

[Schlichting, 2000]
**e^N model**

**Initial perturbation:** \( a_0 = ? \)

**Growth factor:** \( n(x, \beta) = \ln\left(\frac{a(x, \beta)}{a_0}\right) = \int_{x_0}^{x} -\alpha_i(\beta) \, d\xi. \)

**Transition location:** \( \max_{\beta} n(x_{tr}, \beta) = n_{crit} \approx 9. \)

van Ingen (\( Tu > 0.1\% \)):
- transition start: \( n_1 = 2.13 - 6.18 \log(Tu) \)
- transition complete: \( n_2 = 5 - 6.18 \log(Tu) \)

**Envelope method [Drela]:**

\[
N = \frac{dN}{dRe_\theta} (Re_\theta - Re_{\theta, crit}),
\]
\[
\frac{dN}{dRe_\theta} = f(H),
\]
\[
\log_{10}(Re_{\theta, crit}) = 2.492 \left( \frac{1}{H - 1} \right)^{0.43} + 0.7 \left( \tanh\left( 14 \frac{1}{H - 1} - 9.24 \right) + 1 \right)
\]
e^N model

Pros:
- Simple and reasonably accurate model for natural transition
- Built on mathematical background (Orr-Sommerfeld eq.)
- Easy to couple with integral boundary layer methods and inviscid models (see e.g. Xfoil package)
- Very useful for (2D) flows with low Tu (wind-turbines, sailplanes, ...)

Cons:
- Does not cover bypass transition
- Difficult to extend the model to complex 3D flows
- Not directly compatible with general N-S codes
Intermittency based models

Intermittency: \[ \gamma(x) := \lim_{T \to \infty} \frac{1}{T} \int_0^T I(t) \, dt \]

For transition in boundary layer [Narasimha, Emmons]:

\[ \gamma(x) = 1 - \exp \left( \frac{N \sigma}{U} (x - x_t)^2 \right) = 1 - \exp \left( \hat{N} \sigma (Re_x - Re_{xt})^2 \right). \]

Correlations for \( N, \sigma \), eg. [Mayle], [Gostelow]

\[ \hat{N} \sigma = 15 \cdot 10^{-12} Tu^{7/4} F(K) \]

Correlations for \( x_t \) (\( Re_{xt} \)), eg. [Mayle], [Abu-Ghannam, Shaw], ...

\[ Re_{\theta t} = 400 Tu^{-0.625}, \quad \text{for } Tu > 1\% \]

\[ Re_{\theta t} = 163 + \exp \left( F(\lambda_\theta) - \frac{F(\lambda_\theta)}{6.91} Tu \right), \quad \text{for low } Tu \]

\[ Re_\theta = \frac{U \theta}{\nu}, \quad \lambda_\theta = \frac{\theta^2}{\nu} \frac{dU}{ds}. \]
Algebraic intermittency model

Model by Příhoda & Straka:

- Calculation of $Re_\theta$ via $Re_{v,\text{max}}$ assuming non-zero pressure gradient
- Improved correlations

Coupling to a two-equation $k$-$\omega$ model:

\[
\begin{align*}
\frac{Dk}{Dt} &= \tilde{P}_k - \tilde{D}_k + \nabla (\nu_k^{\text{eff}} \nabla k), \\
\frac{D\omega}{Dt} &= P_\omega - D_\omega - C_D + \nabla (\nu_\omega^{\text{eff}} \nabla \omega), \\
\tilde{P}_k &= \gamma P_k, \\
\tilde{D}_k &= \max(\gamma, 0.1) D_k, \\
\nu^{\text{eff}} &= \nu + \gamma \nu_t.
\end{align*}
\]

\[
\begin{align*}
\gamma &= \frac{\gamma_i + \gamma_c}{2} + \frac{\gamma_e - \gamma_i}{2} \tanh \left[ C_\gamma \left( \frac{d}{\delta_{995}} - 1 \right) \right], \\
\gamma_i &= \begin{cases} 
1 - \exp[\hat{n}\sigma (Re_x - Re_{xt})^2], & \text{for } Re_x > Re_{xt} \\
0, & \text{otherwise.}
\end{cases}
\end{align*}
\]

\[
\begin{align*}
Re_{\theta t} &= Re_{\theta 0} \left[ 1 + F(Tu) \frac{1 - \exp(-40\lambda_t)}{1 + 0.4 \exp(-40\lambda_t)} \right], \\
Re_{\theta 0} &= \begin{cases} 
975.8 - 497.2 Tu + \frac{11.4}{Tu} & \text{for } Tu \leq 1\% , \\
96.7 + \frac{340}{Tu} + \frac{53.3}{Tu^2} & \text{for } Tu > 1\%.
\end{cases}
\end{align*}
\]

\[
F(Tu) = 0.29[1 - 0.54 \exp(-3.5 Tu)] \exp(-Tu),
\]

\[
N = \begin{cases} 
0.86 \times 10^{-3} Tu^{-0.564} \exp(2.134 \lambda_t \ln Tu - 59.23 \lambda_t), & \lambda_t < 0, \\
0.86 \times 10^{-3} Tu^{-0.564} \exp(-10\sqrt{\lambda_t}), & \lambda_t \geq 0.
\end{cases}
\]

\[
N = \hat{n}\sigma Re_{\theta t}^3
\]

\[
K = \frac{\nu dU e(s)}{U_e^2 ds},
\]

\[
L = Re_{v,\text{max}}^2 K,
\]

\[
C = 2.185 - 5.79 L + 63.076 \min(0, L)^4,
\]

\[
Re_\theta = \frac{Re_{v,\text{max}}}{C},
\]

\[
\lambda = Re_\theta^2 K,
\]
Algebraic intermittency model

Flat plate flows [Fürst, Příhoda, Straka: Computing, 2013]

ERCOFTAC T3A case
Tu=3%

ERCOFTAC T3A- case
Tu=0.9%

ERCOFTAC T3B case
Tu=6%

Flows through a turbine cascade [FPS, 2013]
Pros:
- Able to cover natural and bypass transition
- Easy to test/implement custom correlations
- Extensible for other scenarios of the laminar-turbulent transition (transition in separated flows, ...)
- Computationally inexpensive (for simple geometry & structured meshes)

Cons:
- Very difficult implementation in the case of unstructured meshes
- Basically 2D model
- Does not reflect the history of the flow
Concept of the model:

\[ \gamma(x) = 1 - \exp\left(-\beta_\gamma^2(x - x_t)^2\right) \quad \rightarrow \quad \sqrt{-\ln(1 - \gamma)} = \beta_\gamma(x - x_t) \]

\[ \vec{u} \cdot \nabla \gamma = 2\beta_\gamma \|\vec{u}\|(1 - \gamma)\sqrt{-\ln(1 - \gamma)} \quad \leftarrow \quad \frac{d\gamma}{dx} = 2\beta_\gamma(1 - \gamma)\sqrt{-\ln(1 - \gamma)} \]

\[ \frac{D\gamma}{Dt} = 2\sqrt{\hat{N}\sigma} \frac{U\|u\|}{\nu}(1 - \gamma)\sqrt{-\ln(1 - \gamma)}F_{onset} + \nabla(\nu^e_{\gamma} \nabla \gamma) \]

\[ \frac{D\gamma}{Dt} = F_{length}S(1 - \gamma)\sqrt{\gamma}F_{onset} + \nabla(\nu^e_{\gamma} \nabla \gamma), \text{ where } S = \sqrt{2S_{ij}S_{ij}} \]
Intermittency transport model

Langtry and Menter, 2006 and 2009:

\[
\frac{D \gamma}{Dt} = P_\gamma - E_\gamma + \frac{\partial}{\partial x_j} \left[ \left( \nu + \frac{\nu_t}{\sigma_\gamma} \right) \frac{\partial \gamma}{\partial x_j} \right]
\]

\[
P_\gamma = c_{a1} F_{\text{length}} S (1 - c_{e1} \gamma) \sqrt{\gamma F_{\text{onset}}},
\]

\[
E_\gamma = c_{a2} F_{\text{turb}} \Omega (c_{e2} \gamma - 1) \gamma
\]

\[
F_{\text{turb}} = F_{\text{turb}}(\nu_t/\nu)
\]

From Blasius profile:

\[
Re_\theta = \frac{\max_y Re_\nu}{2.193}, \quad Re_\nu = \frac{y^2 S}{\nu}
\]

\[
F_{\text{onset}} = F_{\text{onset}} \left( \frac{Re_\nu}{2.193Re_{\theta c}}, \frac{\nu_t/\nu}{\nu} \right)
\]

Correlation for \(F_{\text{length}}\) and \(Re_{\theta c}\) [LM 2009]:

\[
\frac{D \bar{Re}_\theta t}{Dt} = P_{\theta t} + \frac{\partial}{\partial x_j} \left[ \sigma_{\theta t} (\nu + \nu_t) \frac{\partial \bar{Re}_\theta t}{\partial x_j} \right]
\]

\[
F_{\text{length}} = F_{\text{length}}(\bar{Re}_\theta t), \quad Re_{\theta c} = Re_{\theta c}(\bar{Re}_\theta t).
\]
Intermittency transport model

Langtry and Menter model coupled with SST model:

Pros:
• Implemented in many software packages
• Possible to change/improve correlations for particular cases (low $Tu$, surface roughness, ...)
• Coupled to well established SST turbulence model

Cons:
• Based on non-physical quantity $Re_{\theta t}$
• Valid for $Tu>0.027\%$, but needs specific tuning of inlet value of $Tu$ and $\omega$ in order to capture some basic test cases

Extensions:
• Model for surface roughness induced transition [Dassler,2010], [Langel et al., 2014]
• Three-equation $\gamma$-SST model of Menter et al., 2015
Laminar kinetic energy model

- Concept by Mayle and Shultz
- Three-equation model based on k-ε [Walters, Leylek]
- Three-equation model based on k-ω [Walters, Cokljat]

1. Laminar flow
2. Tollmien-Schlichting waves ($k_{\Lambda}$)
3. 3D vortices
4. Vortex breakdown
5. Turbulent spots ($k_T$)
6. Fully turbulent flow
Laminar kinetic energy model

\[
\frac{Dk_T}{Dt} = P_{kT} + R_{BP} + R_{NAT} - \omega k_T - D_T + \frac{\partial}{\partial x_j} \left[ \left( \nu + \frac{\alpha_T}{\sigma_k} \right) \frac{\partial k_T}{\partial x_j} \right],
\]

\[
\frac{Dk_L}{Dt} = P_{kL} - R_{BP} - R_{NAT} - D_L + \frac{\partial}{\partial x_j} \left[ \nu \frac{\partial k_L}{\partial x_j} \right],
\]

\[
\frac{D\omega}{Dt} = C_{\omega 1} \frac{\omega}{k_T} P_{kT} + \left( \frac{C_{\omega R}}{f_W} - 1 \right) \frac{\omega}{k_T} (R_{BP} + R_{NAT}) - C_{\omega 2} \omega^2
\]

\[+ C_{\omega 3} f_\omega Q_T f_W^2 \frac{\sqrt{k_T}}{d^3} + \frac{\partial}{\partial x_j} \left[ \left( \nu + \frac{\alpha_T}{\sigma_\omega} \right) \frac{\partial \omega}{\partial x_j} \right].\]

Two length scales of turbulence:
- \( \lambda_{eff} = \min (C_\lambda d, \lambda_T), \lambda_T = \sqrt{k_T/\omega}, k_T,s = f_{SS} (\lambda_{eff}/\lambda_T)^{2/3}. \)
- Small vortices \( \Rightarrow P_{kT}, \) large vortices \( \Rightarrow P_{kL} \)

Energy transfer from \( k_L \) to \( k_T \):
- Natural transition: \( R_{NAT} \) active when \( \text{Re}_\Omega = d^2 \Omega/\nu > C_{NATcrit} \)
- Bypass transition: \( R_{BP} \) active when \( k_T/(\nu \Omega) > C_{BPcrit} \)
Laminar kinetic energy model

ERCOFTAC test cases for flat plate flows

- T3A $Tu = 3\%$
- T3B $Tu = 6\%$
- T3A- $Tu = 0.8\%$

ERCOFTAC T3A case

$Tu=3\%$

ERCOFTAC T3B case

$Tu=6\%$

ERCOFTAC T3A- case

$Tu=0.8\%$
Laminar kinetic energy model

Flows over NACA 0012 profile
- $Re=600\,000$, $Tu=0.3\%$, $AoA=0^\circ$
- Fine mesh with 500x100 cells
- 2$^{nd}$ order FVM method (OpenFOAM)

Comparison with
- Experiment [Lee, Kang]
- XFOil ($e^n$) [Drella]
LKE model for APG flows

- The original Walters and Cokljat model needs a modification for flows with adverse pressure gradient (APG) at low $Tu$.
- New thresholds for $R_{NAT}$ and $P_{KL}$ [Fürst et al., 2015]

Pohlhausen velocity profile:

$$\frac{U(y)}{U_e} = 2\eta - 2\eta^3 + \eta^4 + \frac{\Lambda}{6}\eta(1 - \eta)^3$$

$$\eta = y/\delta, \quad \Lambda = \frac{\delta^2}{\nu} \frac{dU_e}{dx}.$$

Stability limit for Pohlhausen profiles:

- Orr-Sommerfeld eq., [Schlichting, Ulrich, 1942]
- Stability limit $Re_{ind}=f(\Lambda)$
LKE model for APG flows

- Difficult calculation of $\delta$ and $\Lambda$ $\Rightarrow$ reformulated in terms of $Re_\Omega$ and $L$

$$Re_{\Omega, ind} \approx \frac{536.4}{1 - 8.963L}, \text{ for } -1.5 \leq L \leq 0.$$  

$$L = Re_{\Omega, max}^2 \frac{\nu}{U_e^2} \frac{dU_e}{dx}$$

New thresholds for APG flows:

- Threshold for $P_{kl}$:  
  $$C_{TS, crit}^{APG} = Re_{\Omega, ind} = \frac{536.4}{1 - 8.963L}$$

- Threshold for $R_{NAT}$:  
  $$C_{NAT, crit} = \frac{1250}{1 - 8.963L}.$$
LKE model for APG flows

Flows over NACA 0012 profile, Re=600 000, Tu=0.3%
LKE model for APG flows

Flows over NACA 0012 profile, Re=600 000, Tu=0.3%
LKE model for APG flows

Flows through the VKI cascade, Re=2 110 000, $M_{2i} = 1.089$ (case 241)
• Experimental data by Arts
• Fixed wall temperature, measurements of heat flux
Transitional flow in 3D case

Flows through 3D prismatic cascade (TR-L-1), Re=1 200 000, $M_{2i}=1.2$

- Experimental data by IT CAS

Span-wise distribution of kinetic energy loss
Conclusions

- Laminar-turbulent transition plays an important role in accurate prediction of flows in many engineering applications
- There exist a lot of ways from laminar to turbulent boundary layer (natural transition, bypass transition, transition in LSB, ...)
- There is no model capturing all transition scenarios!
- Existing RANS based transition models heavily rely on the experimental data via correlations.

Future of RANS based transition models

- DNS – still too expensive
- LES – for proper transition modeling on needs to resolve small scale fluctuations in the vicinity of the wall => almost as expensive as DNS
- Hybrid RANS-LES (DES) – combination of RANS model in the vicinity of the wall with LES in freestream => necessity of RANS based model.

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