From local to global theories of iteration

Václav Kučera

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Newton’s method

Iteration of quadratic functions

Mandelbrot set

Extension to \( \mathbb{R}^n \)
1. Newton’s method
2. Iteration of quadratic functions
3. Mandelbrot set
4. Extension to $\mathbb{R}^n$
Newton’s method

- Newton’s method for $x^2 = A$:

$$x_{k+1} = \frac{1}{2} \left( x_k + \frac{A}{x_k} \right)$$

- “Babylonian method”, “Hero’s method”, 1st century CE.
- Converges for any $x_0 \neq 0$ to $\pm \sqrt{A}$.
- Cayley’s problem: What happens for equations in $\mathbb{C}$?

Cayley’s theorem (1879)

Starting from $z_0 \in \mathbb{C}$, the iterates of Newton’s method for $z^2 = c$ converge to the root closer to $z_0$.

- “The next succeeding case of the cubic equation appears to present considerable difficulty.”
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Newton’s method for $z^3 = 1$
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- Fractal basins of attraction.
- **Wada property:** Any neighborhood of any point on the boundary intersects all three basins.

Basins = Fatou set, boundary = Julia set
Chaotic behavior on Julia set, sensitive dependence in its neighborhood.
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![Fractal basins](image)

- Basins = **Fatou set**, boundary = **Julia set**
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Newton’s method for $z^8 + 15z^4 - 16$
Newton’s method for $z^3 - 2z + 2$
In 1915, The Paris Academy of Sciences announced a “Great Prize of mathematical sciences for the year 1918”.

The topic is the behavior of the iterates $P_n = \varphi(P_{n-1})$.

“Up to now, the well known works devoted to this investigation are mainly about the ‘local’ point of view. The Academy considers that it would be interesting to proceed from here to the examination of the whole domain of the values taken by the variables”.

100 years later, numerical mathematics still has mostly local theories (ball convergence).

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Kantorovich theorem
Newton’s method

Iteration of quadratic functions

Mandelbrot set

Extension to $\mathbb{R}^n$

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Newton’s method for $z^2 = 1$ converges to the root closer to $z_0$.

Proof:

- We iterate $z_{k+1} = N(z_k)$, where $N(z) = \frac{1}{2} (z + \frac{1}{z})$.
- Let $\varphi(z) = \frac{z - 1}{z + 1}$.
- Then $\varphi \circ N \circ \varphi^{-1}(z) = f(z)$, where $f(z) = z^2$.
- $\varphi \circ N^0 \circ \varphi^{-1}(z) = f^0(z)$.

\[\begin{array}{ccc}
\mathbb{C} & \xrightarrow{\varphi} & \mathbb{C} \\
N & \downarrow & f(z) = z^2 \\
\mathbb{C} & \xrightarrow{\varphi} & \mathbb{C} \\
\end{array}\]

$\varphi(1) = 0$,

$\varphi(-1) = \infty$,

$\varphi(\{\text{Re}z = 0\}) = \{|z| = 1\}$
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1. Newton’s method
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4. Extension to $\mathbb{R}^n$
Let $z_0 \in \mathbb{C}$ and $z_{n+1} = z_n^2$. What happens as $n \to +\infty$?

- If $z = re^{2\pi i \alpha}$, then $z^2 = r^2e^{2\pi i (2\alpha)}$.

  $|z_0| \begin{cases} < 1 \text{ then } z_n \to 0, \\ > 1 \text{ then } z_n \to \infty. \end{cases}$

- What happens if $|z_0| = 1$?
- If $z_n = e^{2\pi i \alpha_n}$ then $z_{n+1} = z_n^2 = e^{2\pi i (2\alpha_n)}$, therefore

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  \begin{align*}
  \{ |z| = 1 \} & \xleftarrow{e^{2\pi i \alpha} \leftarrow \alpha} [0, 1) \\
  z \mapsto z^2 & \downarrow \alpha \mapsto 2\alpha \pmod{1} \\
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Iteration of $z^2$ in $\mathbb{C}$

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$$\begin{cases} |z| = 1 & \xrightarrow{e^{2\pi i \alpha} \leftarrow \alpha} \mathbb{R} \cap [0,1) \quad \downarrow \quad \alpha \rightarrow 2\alpha \mod 1 \\ z \rightarrow z^2 & \quad \downarrow \quad \alpha \rightarrow 2\alpha \mod 1 \\ \{ |z| = 1 \} & \xrightarrow{e^{2\pi i \alpha} \leftarrow \alpha} \mathbb{R} \cap [0,1) \end{cases}$$
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$\alpha \mapsto 2\alpha \pmod{1}$
Writing $\alpha$ in binary form, then e.g.

$$\alpha = 0.11010011101\ldots$$

$$2\alpha \pmod{1} = 0.1010011101\ldots$$

Bernoulli shift

Define the shift map $\sigma : \{0,1\}^\mathbb{N} \rightarrow \{0,1\}^\mathbb{N}$ by

$$\sigma(s) = \sigma(s_0s_1s_2\ldots) = (s_1s_2s_3\ldots).$$

The dynamics of $z^2$ is conjugate to that of $\sigma$. 
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Examples

Period two:

- Take $s = (010101\ldots)$, then

  $$\sigma(s) = (101010\ldots),$$
  $$\sigma(\sigma(s)) = s.$$

- Corresponds to $\alpha = 0.010101\ldots = 1/3$, then

  $$2\alpha \pmod{1} = 2/3,$$
  $$4\alpha \pmod{1} = 1/3.$$

- Corresponds to

  $$z = e^{2\pi i/3} = \frac{1}{2}(-1 + \sqrt{3}i),$$
  $$z^2 = e^{2\pi i(2/3)} = \frac{1}{2}(-1 - \sqrt{3}i),$$
  $$(z^2)^2 = z.$$
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Examples

Period 3, #1:
- Take \( s = (001001\ldots) \), then \( \sigma^3(s) = s \).
- Corresponds to \( \alpha = \frac{1}{7} \) and the sequence
  \[
  \begin{array}{cccc}
  1 & 2 & 4 & 1 \\
  7 & 7 & 7 & 7 \\
  \end{array}
  \]
- Corresponds to \( z = e^{2\pi i/7} \), then \( ((z^2)^2)^2 = z \).

Period 3, #2:
- Take \( s = (011011\ldots) \), then \( \sigma^3(s) = s \).
- Corresponds to \( \alpha = \frac{3}{7} \) and the sequence
  \[
  \begin{array}{cccc}
  3 & 6 & 5 & 3 \\
  7 & 7 & 7 & 7 \\
  \end{array}
  \]
- Corresponds to \( z = e^{2\pi i(3/7)} \), then \( ((z^2)^2)^2 = z \).
Examples

Period 3, #1:
- Take \( s = (001001 \ldots) \), then \( \sigma^3(s) = s \).
- Corresponds to \( \alpha = \frac{1}{7} \) and the sequence
  \[
  1 \rightarrow 2 \rightarrow 4 \rightarrow 1 \rightarrow 7
  \]
  \( \frac{1}{7} \rightarrow \frac{2}{7} \rightarrow \frac{4}{7} \rightarrow \frac{1}{7} \)
- Corresponds to \( z = e^{2\pi i/7} \), then \( ((z^2)^2)^2 = z \).

Period 3, #2:
- Take \( s = (011011 \ldots) \), then \( \sigma^3(s) = s \).
- Corresponds to \( \alpha = \frac{3}{7} \) and the sequence
  \[
  3 \rightarrow 6 \rightarrow 5 \rightarrow 3 \rightarrow 7
  \]
  \( \frac{3}{7} \rightarrow \frac{6}{7} \rightarrow \frac{5}{7} \rightarrow \frac{3}{7} \)
- Corresponds to \( z = e^{2\pi i(3/7)} \), then \( ((z^2)^2)^2 = z \).
Examples

Period 3, #1:

- Take \( s = (001001\ldots) \), then \( \sigma^3(s) = s \).
- Correlates to \( \alpha = 1/7 \) and the sequence
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Period 3, #2:

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  3 \rightarrow 6 \rightarrow 5 \rightarrow 3 \rightarrow 7 \rightarrow 7 \rightarrow 7 \rightarrow 7
  \]

- Correlates to \( z = e^{2\pi i(3/7)} \), then \( (z^2)^2 = z \).
Examples

Period 3, #1:

- Take $s = (001001\ldots)$, then $\sigma^3(s) = s$.
- Corresponds to $\alpha = 1/7$ and the sequence
  
  \[
  \begin{align*}
  1 &\rightarrow 2 &\rightarrow 4 &\rightarrow 1 \\
  \overline{7} &\rightarrow \overline{7} &\rightarrow \overline{7} &\rightarrow \overline{7}
  \end{align*}
  \]

- Corresponds to $z = e^{2\pi i/7}$, then $((z^2)^2)^2 = z$.

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(Pre)periodic points

Lemma

Let $\alpha = \frac{a}{2^p - 1}$ for $p \in \mathbb{N}$ and $a \in \{1, \ldots, 2^p - 2\}$. Let $z_0 = e^{2\pi i \alpha}$ and $z_{n+1} = z_n^2$, then $\{z_n\}_{n \in \mathbb{N}}$ has period $p$.

Corollary

Periodic points of $z \mapsto z^2$ are dense in $\{|z| = 1\}$.

Preperiodic points: 0.11010011101001001...
Point with dense orbit:

\[
0. \quad \underbrace{01}_{1\text{-blocks}} \quad \underbrace{00\ 01\ 10\ 11}_{2\text{-blocks}} \quad \underbrace{000\ 001}_{3\text{-blocks}} \quad \ldots \quad \ldots \quad \ldots
\]

Corresponds to
\[
z = e^{2\pi i 0.27638711728\ldots} \approx -0.1650366 + 0.9862874i
\]

Václav Kučera

From local to global theories of iteration
**Lemma**

Let $\alpha = \frac{a}{2^p-1}$ for $p \in \mathbb{N}$ and $a \in \{1, \ldots, 2^p - 2\}$. Let $z_0 = e^{2\pi i \alpha}$ and $z_{n+1} = z_n^2$, then $\{z_n\}_{n \in \mathbb{N}}$ has **period** $p$.

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\[
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0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

1-blocks 2-blocks 3-blocks 4-blocks

Corresponds to
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Periodic points of \( z \mapsto z^2 \) are dense in \( \{|z| = 1\} \).

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Point with dense orbit:

\[
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From local to global theories of iteration
The mapping $f(z) = z^2$ is chaotic on $\{|z| = 1\}$.

- **Topological transitivity**: For any open subsets $U, V$ there exists $n > 0$ such that $f^n(U) \cap V \neq \emptyset$.

- **Sensitive dependence**: Exists $c > 0$ such that for any $x$ and any $U(x)$ there exists $y \in U(x)$ and $n \in \mathbb{N}$ such that $|f^n(x) - f^n(y)| > c$.

- **Periodic points are dense**.

Summary: Unpredictability, indecomposability, and an element of regularity.

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From local to global theories of iteration
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From local to global theories of iteration
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Summary: Unpredictability, indecomposability, and an element of regularity.
Let $|z_0| = 1$ with some neighborhood $U$. For any $z \in \mathbb{C} \setminus \{0\}$ there exists $n > 0$ such that $z \in f^n(U)$.

- Repelling periodic points are dense in $\{|z| = 1\}$.
- $f^n \Rightarrow 0$ on any compact subset of $\{|z| < 1\}$.
- $f^n \Rightarrow \infty$ on any “compact subset of $\{|z| > 1\}$”.
- $\{|z| = 1\} = \text{Julia set } J(f)$.
- $\{|z| \neq 1\} = \text{Fatou set } F(f)$. 
Iteration of $f(z) = z^2$ in $\overline{\mathbb{C}}$

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Other quadratic polynomials

Figure: $f(z) = z^2 - 1/4$
Other quadratic polynomials

Figure: $f(z) = z^2 - 1$
Other quadratic polynomials

Figure: \( f(z) = z^2 - 2.1 \)
Other quadratic polynomials

Figure: $f(z) = z^2 - 1 + 0.4i$
Other quadratic polynomials

Figure: $f(z) = z^2 + 0.285 + 0.01i$
Iteration of $Q_c(z) = z^2 + c$:

- $J_c$ = closure of the set of repelling periodic points of $Q_c$.
- Let $z_0 \in J_c$ with some neighborhood $U$. Then
  \[
  \bigcup_{n=1}^{\infty} Q_c^n(U)
  \]
  omits at most one point in $\mathbb{C}$.
- $J_c$ has empty interior.
- $J_c \neq \emptyset$.
- $J_c$ is fractal, except for $c = 0$ and $c = -2$.
- Exists $c$ such that $J_c$ has positive area (Buff, Chéritat: 2012).
- $\{Q^n_c\}_{n \in \mathbb{N}}$ is normal on $\mathbb{C} \setminus J_c$: there exists a subsequence converging locally uniformly on $\overline{\mathbb{C}} \setminus J_c$. 

Properties - Julia, Fatou ($\sim$1917)
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\[
\bigcup_{n=1}^{\infty} Q_c^o(U)
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Newton’s method

Iteration of quadratic functions

Mandelbrot set

Extension to $\mathbb{R}^n$

Properties - Julia, Fatou ($\sim$1917)

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Václav Kučera

From local to global theories of iteration
1. Newton’s method
2. Iteration of quadratic functions
3. Mandelbrot set
4. Extension to $\mathbb{R}^n$
Mandelbrot set

Filled-in Julia set

\[ K_c = \{ z; \ Q_c^n(z) \not\to \infty \} \]

Lemma

If \( Q_c^n(0) \not\to \infty \) then \( K_c \) is connected. Else \( K_c \) is a Cantor set.

Mandelbrot set

\[ \mathcal{M} := \{ c; \ K_c \text{ is connected} \} \]
Mandelbrot set

**Filled-in Julia set**

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Mandelbrot set

\[ \mathcal{M} := \{ c; \ K_c \text{ is connected} \} \]
Mandelbrot set (1980)
Brooks, Matelski (1978) *

Lemma - Uniformization

Let $K_c$ be connected. Then there exists an analytic homeomorphism $\Phi_c : \mathbb{C} \setminus K_c \to \{|z| > 1\}$ such that $\Phi_c(Q_c(z)) = (\Phi_c(z))^2$.

$$
\begin{align*}
\mathbb{C} \setminus K_c & \xrightarrow{\Phi_c} \{|z| > 1\} \\
Q_c & \downarrow \quad \downarrow z^2 \\
\mathbb{C} \setminus K_c & \xrightarrow{\Phi_c} \{|z| > 1\}
\end{align*}
$$
External rays - Douady, Hubbard (1982)
\[
\frac{6k - 1}{3 \cdot 2^n} \sim \frac{6k + 1}{3 \cdot 2^n}
\]
Laminations - Douady rabbit

\[ z^2 - 0.123 + 0.745i \]
Lemma

The mapping $\Phi(c) = \Phi_c(c)$ is an analytic homeomorphism of $\mathbb{C} \setminus M$ onto $\{ |z| = 1 \}$.

Corollary

$\mathcal{M}$ is connected.
Uniformization of $M$

**Lemma**

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**Corollary**

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External rays
Lavaurs’ algorithm (1986)

- Connect 1/3 and 2/3.
- For \( p = 3, 4, \ldots \) connect angles \( n/(2^p - 1) \) s.t.:
  - No arcs cross.
  - Start from \( 1/(2^p - 1) \).
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Topological model of $M$. Correct if MLC holds.
Farrey addition

\[ \frac{1}{3} \oplus \frac{2}{5} = \frac{3}{8}. \]
Further properties

- Small copies of $\mathcal{M}$ are **dense** in $\partial \mathcal{M}$.
- $\dim_H \partial \mathcal{M} = 2$.
- Does $\partial \mathcal{M}$ have positive area?
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![Mandelbrot set and periodic cycles](image-url)
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![Mandelbrot set diagram](image-url)
Mandelbrot locally connected (MLC) conjecture

**Conjecture**

\( \mathcal{M} \) is locally connected.

**Implications:**
- If \( c \in \text{int}(\mathcal{M}) \) then \( J_c \) has an attracting periodic cycle.
- Same for \( \mathcal{M} \cap \mathbb{R} \) (known to hold).
- The “pinched disk” model above is correct.
- External rays “land” on \( \partial \mathcal{M} \) (known for some).

**Challenge question**

Does \( f(z) = z^2 - 1.99999 \) have an attracting periodic cycle?

McMullen: “It is unlikely this question will ever be rigorously settled.” For \( z^2 - c \):
- \( \{ c; \text{answer is no} \} \) is dense at \( c = -2 \).
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1. Newton’s method
2. Iteration of quadratic functions
3. Mandelbrot set
4. Extension to $\mathbb{R}^n$
Quaternions

\[
(x, y, z, w)^2 = (x^2 - y^2 - z^2 - w^2, 2xy, 2xz, 2yw)
\]
Tetrabrot: Bicomplex numbers

$$(x, y, z, w)^2 = (x^2 - y^2 - z^2 + w^2, 2(xy - zw), 2(xz - yw), 2(xw + yz))$$
Extension to $\mathbb{R}^n$

Mandelbulb

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From local to global theories of iteration
Möbius transformations

\[ z \mapsto \frac{az+b}{cz+d} \] maps circles to circles.
Kleinian limit sets
Kleinian limit sets

From local to global theories of iteration
Sullivan’s dictionary

Kleinian limit sets

Julia sets

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Inflated Julia sets

\[ z_{n+1} = z_n^2 - 1 \]
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\[ z_{n+1} = z^2_n - 1 \]
Inflated Julia sets

$$z_{n+1} = z_n^2 - 0.123 + 0.75i$$
Preliminary 3D Julia sets

\[ z_{n+1} = z_n^2 - 1 \]
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Thank you for your attention