From local to global theories of iteration

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1. Newton's method
2. Iteration of quadratic functions
3. Mandelbrot set
4. Extension to $\mathbb{R}^n$

1 Newton's method

2 Iteration of quadratic functions

3 Mandelbrot set

4 Extension to $\mathbb{R}^n$

Newton's method

- Newton's method for $x^2 = A$:

$$x_{k+1} = \frac{1}{2}\left(x_k + \frac{A}{x_k}\right)$$

- "Babylonian method", "Hero's method", 1st century CE.
- Converges for any $x_0 \neq 0$ to $\pm\sqrt{A}$.
- Cayley's problem: What happens for equations in $\mathbb{C}$?

Cayley's theorem (1879)

Starting from $z_0 \in \mathbb{C}$, the iterates of Newton's method for $z^2 = c$ converge to the root closer to z_0.

- "The next succeeding case of the cubic equation appears to present considerable difficulty."

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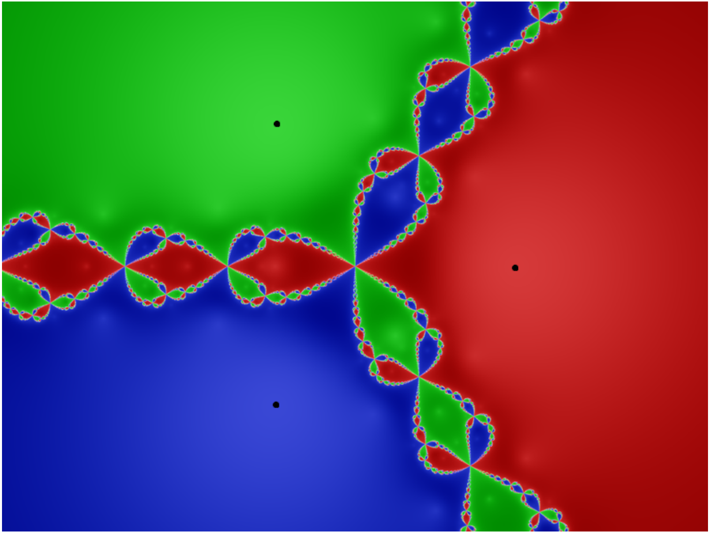

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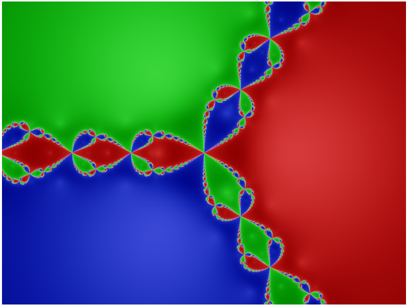
- Fractal basins of attraction.
- Wada property: Any neighborhood of any point on the boundary intersects all three basins.



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- Chaotic behavior on Julia set, sensitive dependence in its neighborhood.

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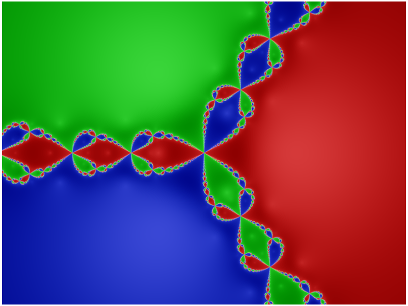

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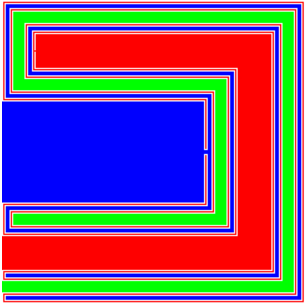

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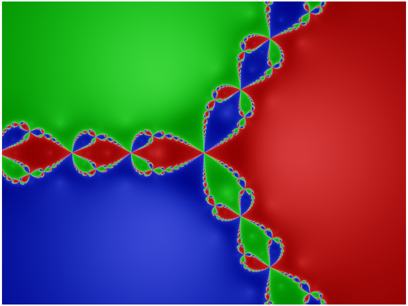

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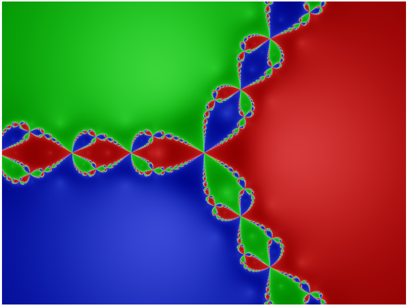

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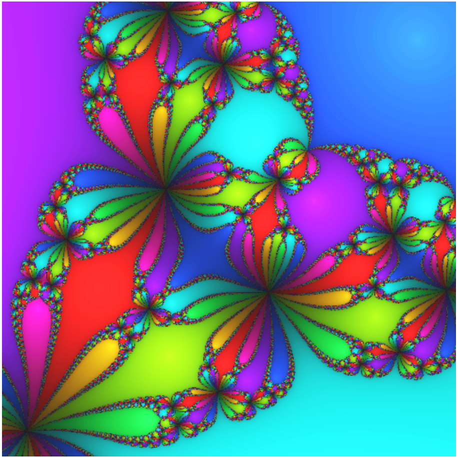
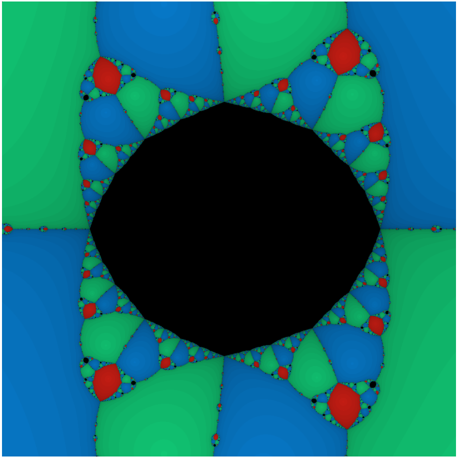
Newton's method for $z^8 + 15z^4 - 16$



Newton's method for $z^3 - 2z + 2$



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- The topic is the behavior of the iterates $P_n = \varphi(P_{n-1})$.
- "Up to now, the well known works devoted to this investigation are mainly about the 'local' point of view. The Academy considers that it would be interesting to proceed from here to the examination of the whole domain of the values taken by the variables".
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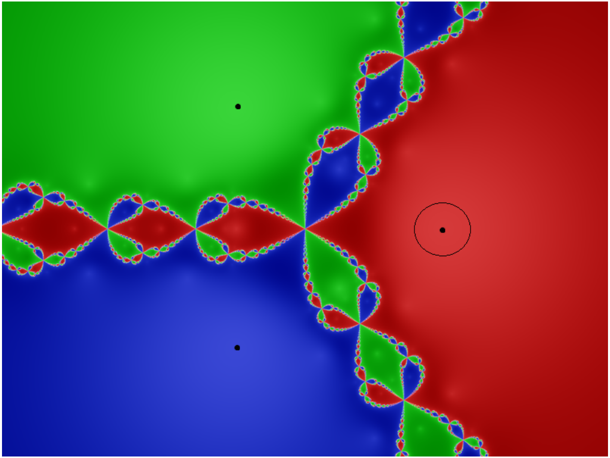
Kantorovich theorem



Cayley's theorem (1879)

Newton's method for $z^2 = 1$ converges to the root closer to z_0.

Proof:

- We iterate $z_{k+1} = N(z_k)$, where $N(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$.
- Let $\varphi(z) = \frac{z-1}{z+1}$.
- Then $\varphi \circ N \circ \varphi^{-1}(z) = f(z)$, where $f(z) = z^2$.
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$$\begin{tikzcd} \overline{\mathbb{C}} \arrow[r, "\varphi"] \arrow[d, "N"'] & \overline{\mathbb{C}} \arrow[d, "f(z)=z^2"] \\ \overline{\mathbb{C}} \arrow[r, "\varphi"] & \overline{\mathbb{C}} \end{tikzcd}$$

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Iteration of z^2 in $\mathbb{C}$

Let $z_0 \in \mathbb{C}$ and $z_{n+1} = z_n^2$. What happens as $n \to +\infty$?

- If $z = r\mathrm{e}^{2\pi\mathrm{i}\alpha}$, then $z^2 = r^2\mathrm{e}^{2\pi\mathrm{i}(2\alpha)}$.

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Bernoulli shift

Writing α in binary form, then e.g.

$$\begin{aligned} \alpha &= 0.11010011101\ldots \\ 2\alpha \pmod 1 &= \ \ 0.1010011101\ldots \end{aligned}$$

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Define the shift map $\sigma : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ by

$$\sigma(\mathbf{s}) = \sigma(s_0 s_1 s_2 \ldots) = (s_1 s_2 s_3 \ldots).$$

The dynamics of z^2 is conjugate to that of σ.

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Examples

Period two:

- Take $\mathbf{s} = (0101\overline{01}\ldots)$, then

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Let $\alpha = \frac{a}{2^p-1}$ for $p \in \mathbb{N}$ and $a \in \{1, \ldots, 2^p-2\}$. Let $z_0 = \mathrm{e}^{2\pi\mathrm{i}\alpha}$ and $z_{n+1} = z_n^2$, then $\{z_n\}_{n\in\mathbb{N}}$ has period p.

Corollary

Periodic points of $z \mapsto z^2$ are dense in $\{|z| = 1\}$.

Preperiodic points: $0.11010011101001\overline{001}\ldots$

Point with dense orbit:

$$0.\underbrace{0\ 1}_{\text{1-blocks}}\ \underbrace{00\ 01\ 10\ 11}_{\text{2-blocks}}\ \underbrace{000\ 001\ \ldots}_{\text{3-blocks}}\ \underbrace{\ldots}_{\text{4-blocks}}\ \ldots$$

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Lemma

The mapping $f(z) = z^2$ is chaotic on $\{|z| = 1\}$.

- Topological transitivity: For any open subsets U, V there exists $n > 0$ such that $f^{\circ n}(U) \cap V \neq \emptyset$.
- Sensitive dependence: Exists $c > 0$ such that for any x and any $U(x)$ there exists $y \in U(x)$ and $n \in \mathbb{N}$ such that $|f^{\circ n}(x) - f^{\circ n}(y)| > c$.
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Summary: Unpredictability, indecomposability, and an element of regularity.

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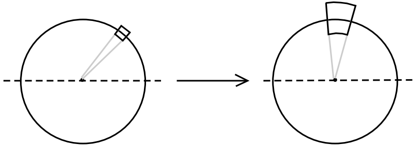
Iteration of $f(z) = z^2$ in $\overline{\mathbb{C}}$



- Let $|z_0| = 1$ with some neighborhood U. For any $z \in \mathbb{C} \setminus \{0\}$ there exists $n > 0$ such that $z \in f^{\circ n}(U)$.
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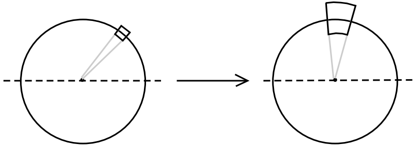

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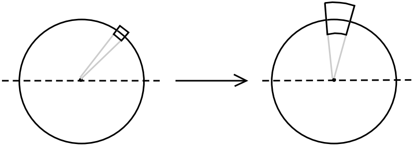

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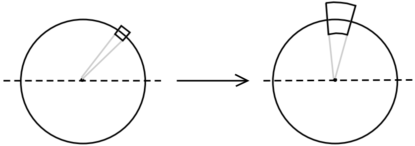

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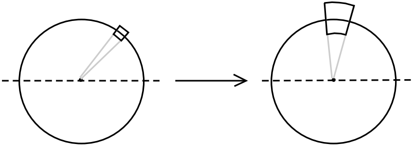

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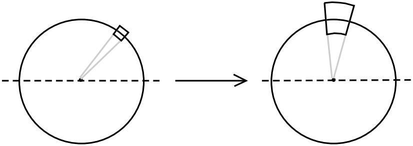

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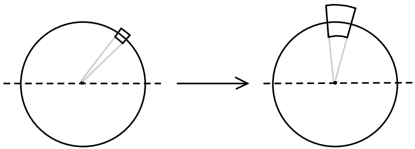

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Other quadratic polynomials

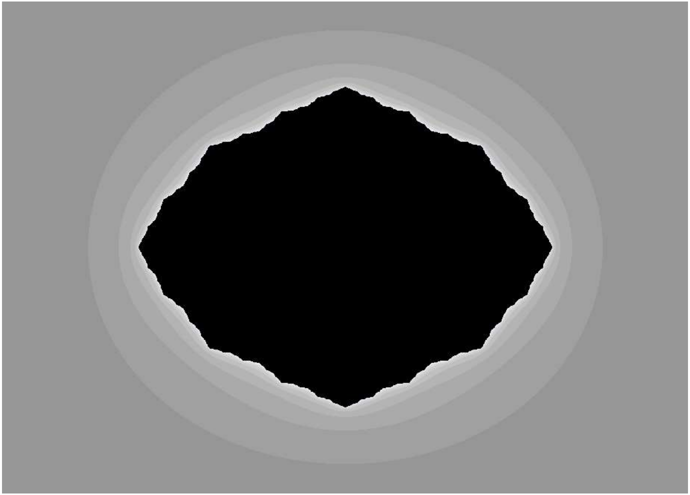

Figure: $f(z) = z^2 - 1/4$

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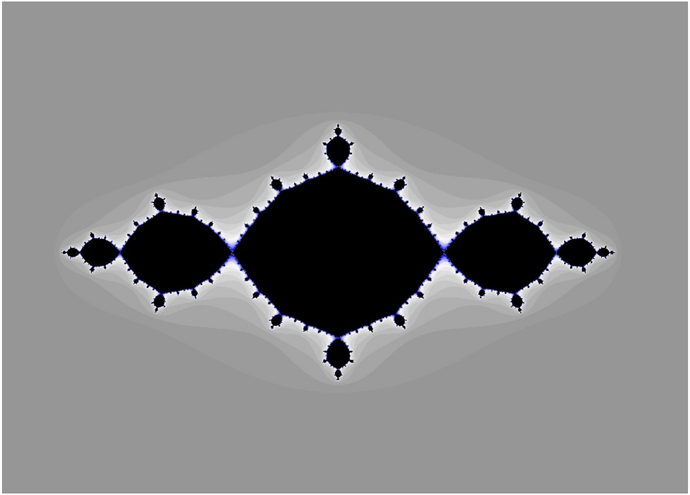

Figure: $f(z) = z^2 - 1$

Other quadratic polynomials

Figure: $f(z) = z^2 - 2.1$

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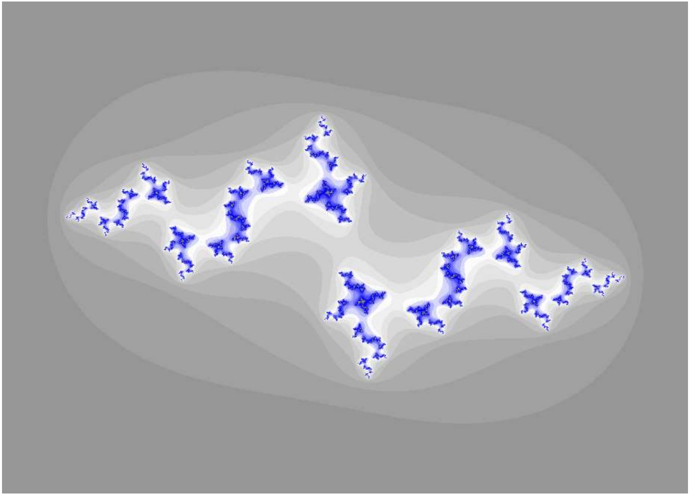

Figure: $f(z) = z^2 - 1 + 0.4\mathrm{i}$

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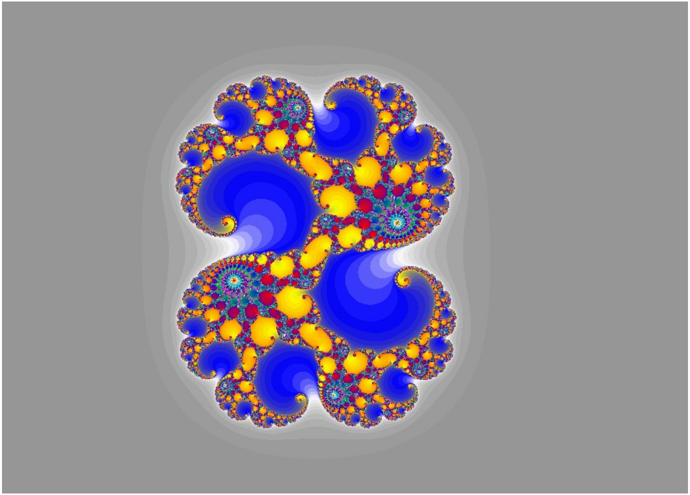

Figure: $f(z) = z^2 + 0.285 + 0.01\mathrm{i}$

Properties - Julia, Fatou ($\sim$1917)

Iteration of $Q_c(z) = z^2 + c$:

- J_c = closure of the set of repelling periodic points of Q_c.
- Let $z_0 \in J_c$ with some neighborhood U. Then

$$\bigcup_{n=1}^{\infty} Q_c^{\circ n}(U)$$

  omits at most one point in $\mathbb{C}$.
- J_c has empty interior.
- $J_c \neq \emptyset$.
- J_c is fractal, except for $c = 0$ and $c = -2$.
- Exists c such that J_c has positive area (Buff, Chéritat: 2012).
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Properties - Julia, Fatou (∼1917)

Iteration of $Q_c(z) = z^2 + c$:

- J_c = closure of the set of repelling periodic points of Q_c.
- Let $z_0 \in J_c$ with some neighborhood U. Then

$$\bigcup_{n=1}^{\infty} Q_c^{\circ n}(U)$$

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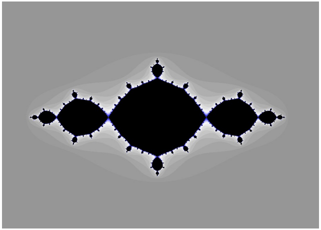
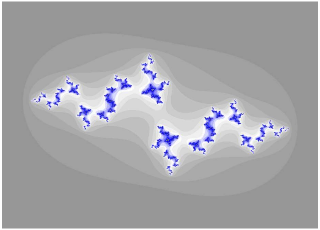
1. Newton's method
2. Iteration of quadratic functions
3. **Mandelbrot set**
4. Extension to $\mathbb{R}^n$

Mandelbrot set

Filled-in Julia set

$$K_c = \{z;\ Q_c^{\circ n}(z) \nrightarrow \infty\}$$





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If $Q_c^{\circ n}(0) \nrightarrow \infty$ then K_c is connected. Else K_c is a Cantor set.

Mandelbrot set

$$\mathscr{M} := \{c;\ K_c \text{ is connected}\}$$

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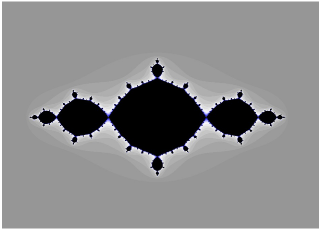

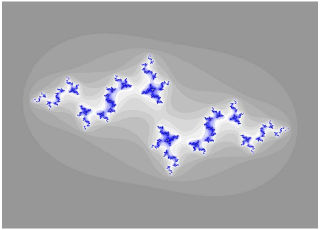

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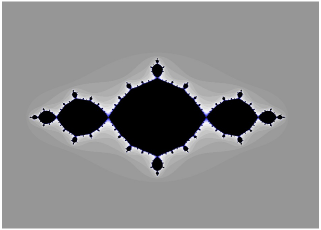
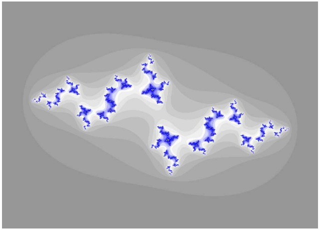

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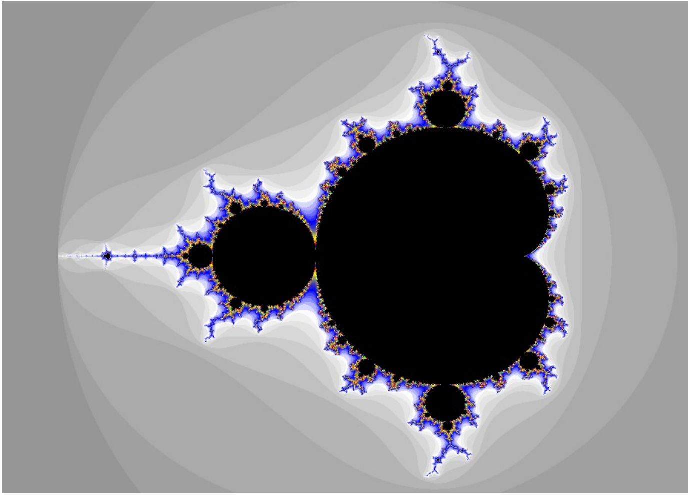
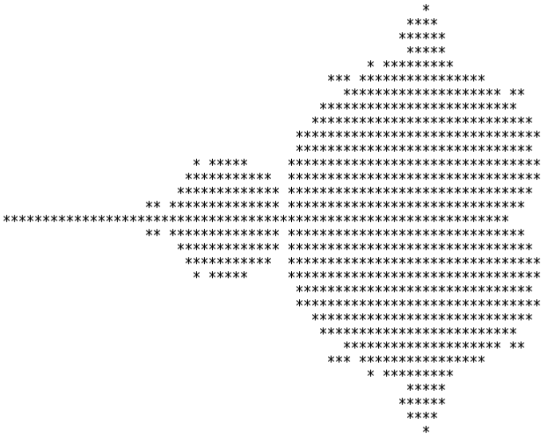
Mandelbrot set (1980)



Brooks, Matelski (1978)

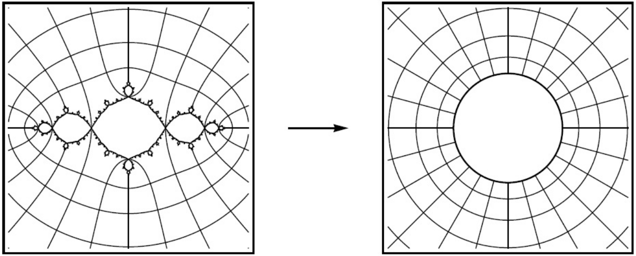


Brooks, Matelski: *The dynamics of 2-generator subgroups of PSL*$(2, \mathbb{C})$, in "Riemann Surfaces and Related Topics", Ann. Math. Stud. 97, 65-71 (1978).

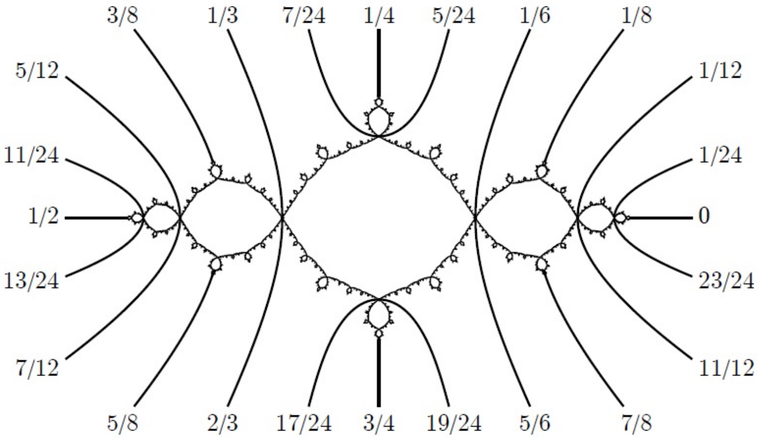
Lemma - Uniformization

Let K_c be connected. Then there exists an analytic homeomorphism $\Phi_c : \mathbb{C} \setminus K_c \to \{|z| > 1\}$ such that $\Phi_c(Q_c(z)) = (\Phi_c(z))^2$.

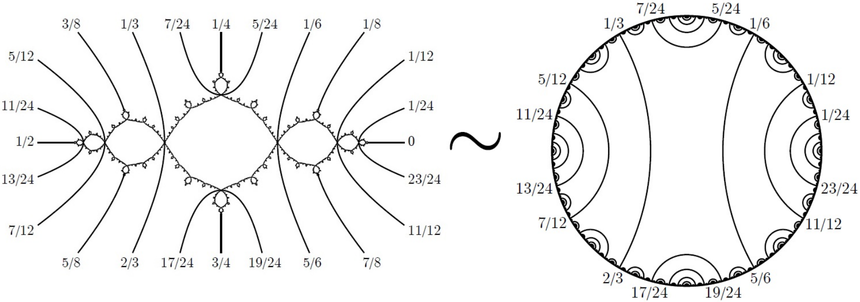
$$\begin{array}{ccc} \mathbb{C} \setminus K_c & \xrightarrow{\Phi_c} & \{|z| > 1\} \\ {\scriptstyle Q_c}\downarrow & & \downarrow{\scriptstyle z^2} \\ \mathbb{C} \setminus K_c & \xrightarrow{\Phi_c} & \{|z| > 1\} \end{array}$$



External rays - Douady, Hubbard (1982)

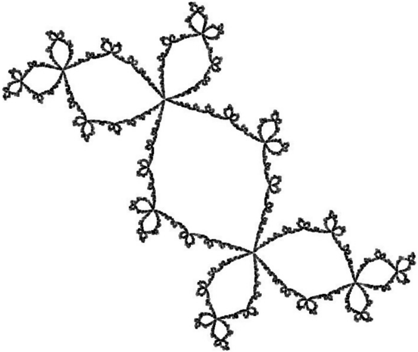


Laminations - Thurston (1985)



$$\frac{6k-1}{3.2^n} \sim \frac{6k+1}{3.2^n}$$

Laminations - Douady rabbit



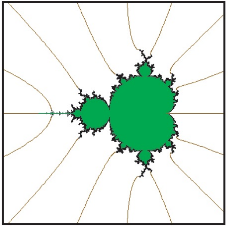
$\sim$

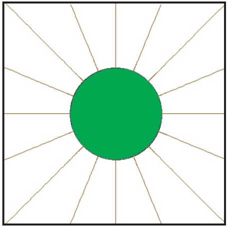
$$z^2 - 0.123 + 0.745i$$

Uniformization of M

Lemma

The mapping $\mathbf{\Phi}(c) = \Phi_c(c)$ is an analytic homeomorphism of $\mathbb{C} \setminus \mathscr{M}$ onto $\{|z| = 1\}$.





Corollary

$\mathscr{M}$ is connected.

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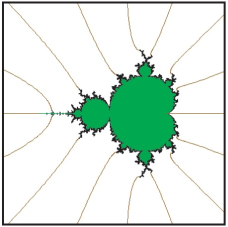

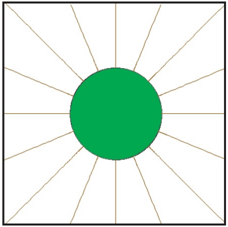

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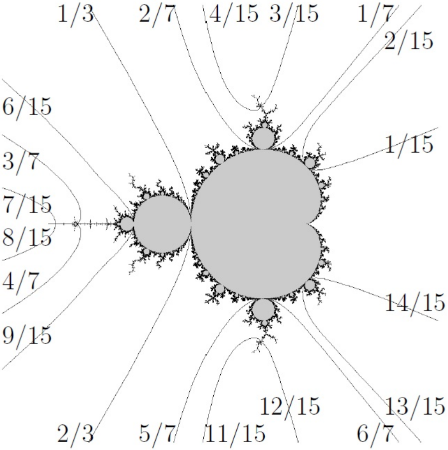

Lavaurs' algorithm (1986)

- Connect $1/3$ and $2/3$.
- For $p = 3, 4, \ldots$ connect angles $n/(2^p - 1)$ s.t.:
  - No arcs cross.
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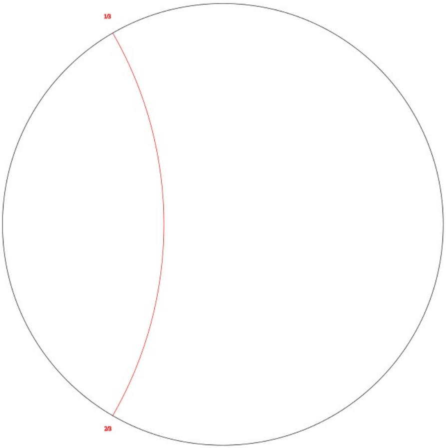

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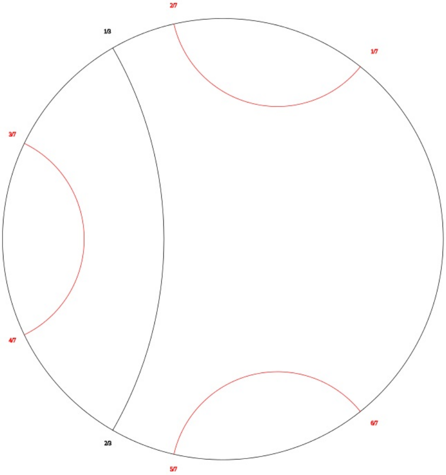

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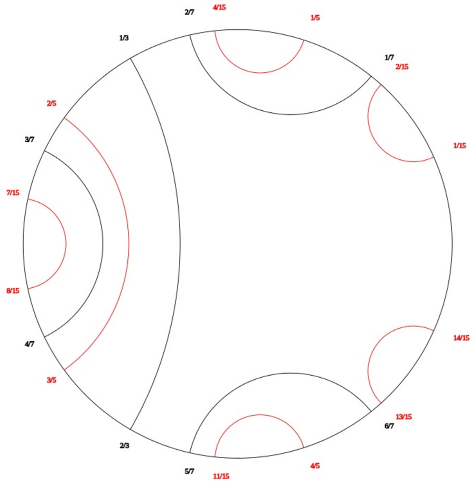

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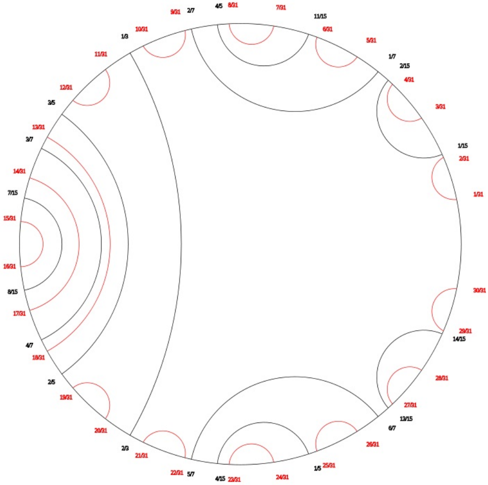
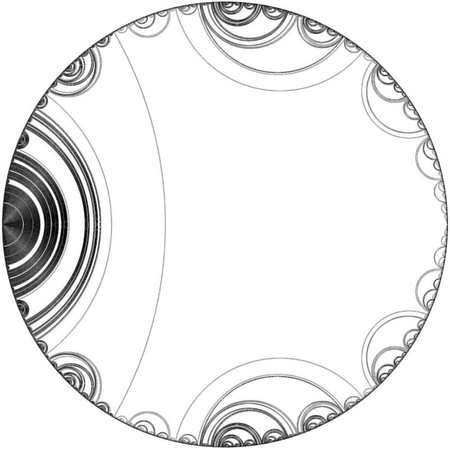
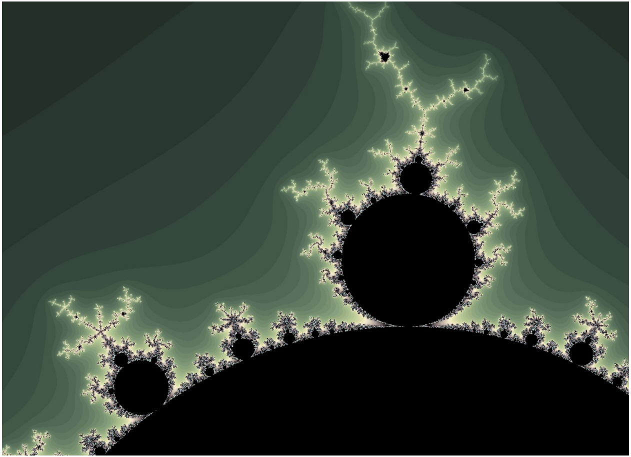


Lamination of $\mathscr{M}$



Topological model of $\mathscr{M}$. Correct if MLC holds.

Farrey addition



$$\frac{1}{3} \oplus \frac{2}{5} = \frac{3}{8}.$$

Further properties

- Small copies of $\mathscr{M}$ are dense in $\partial\mathscr{M}$.
- $\dim_H \partial\mathscr{M} = 2$.
- Does $\partial\mathscr{M}$ have positive area?
- The Mandelbrot set is universal (McMullen, 1997).
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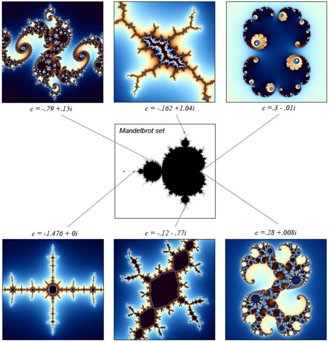

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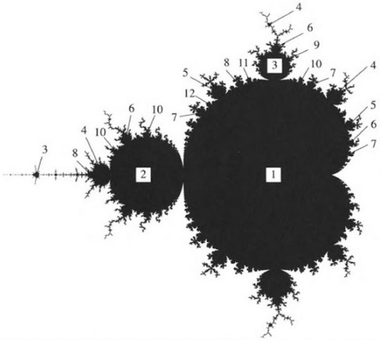

Mandelbrot locally connected (MLC) conjecture

Conjecture

$\mathscr{M}$ is locally connected.

Implications:

- If $c \in \operatorname{int}(\mathscr{M})$ then J_c has an attracting periodic cycle.
- Same for $\mathscr{M} \cap \mathbb{R}$ (known to hold).
- The "pinched disk" model above is correct.
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Challenge question

Does $f(z) = z^2 - 1.99999$ have an attracting periodic cycle?

McMullen: "It is unlikely this question will ever be rigorously settled." For $z^2 - c$:

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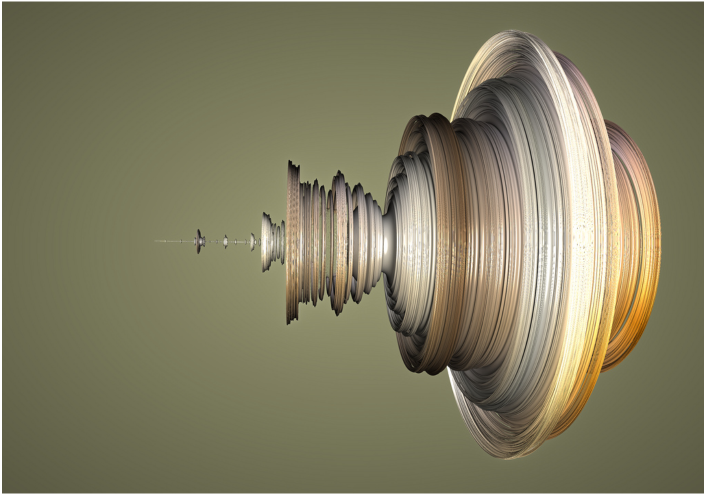
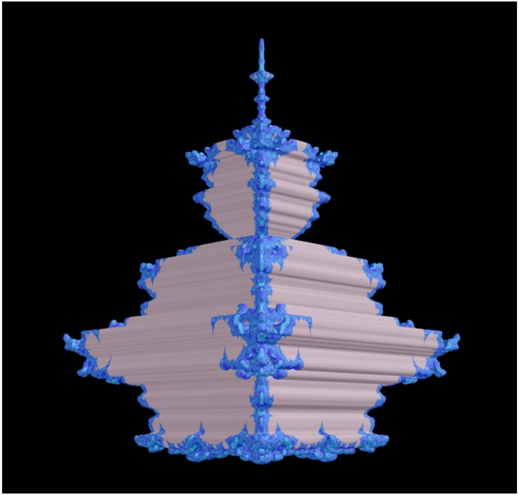


Quaternion Mandelbrot set

$$(x, y, z, w)^2 = (x^2 - y^2 - z^2 - w^2, 2xy, 2xz, 2xw)$$

Extension to $\mathbb{R}^n$



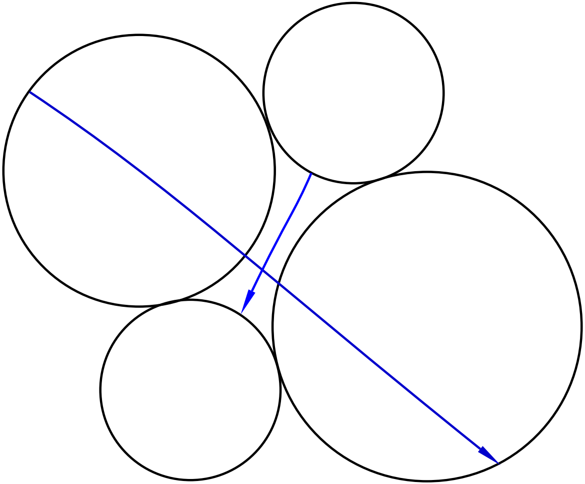
Tetrabrot: Bicomplex numbers

$$(x, y, z, w)^2 = (x^2 - y^2 - z^2 + w^2, 2(xy - zw), 2(xz - yw), 2(xw + yz))$$

Extension to $\mathbb{R}^n$

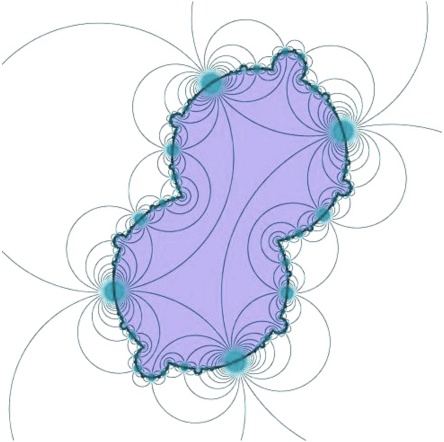
Mandelbulb

Möbius transformations

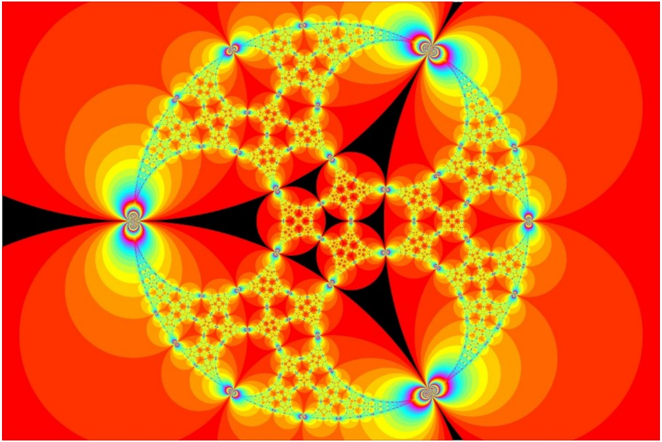


$z \mapsto \frac{az+b}{cz+d}$ maps circles to circles.

Kleinian limit sets



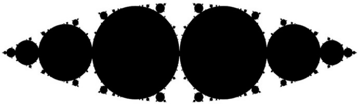
Kleinian limit sets

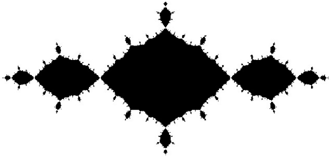


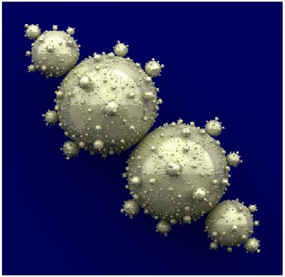
Sullivan's dictionary

Kleinian limit sets

Julia sets



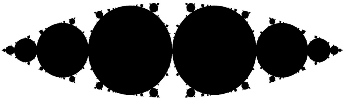




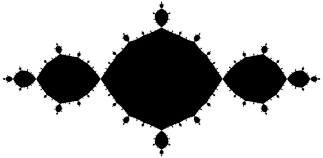
?

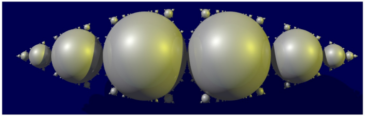
Sullivan's dictionary

Kleinian limit sets | Julia sets



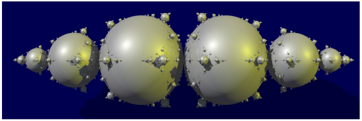
$\longleftrightarrow$





$\longleftrightarrow$

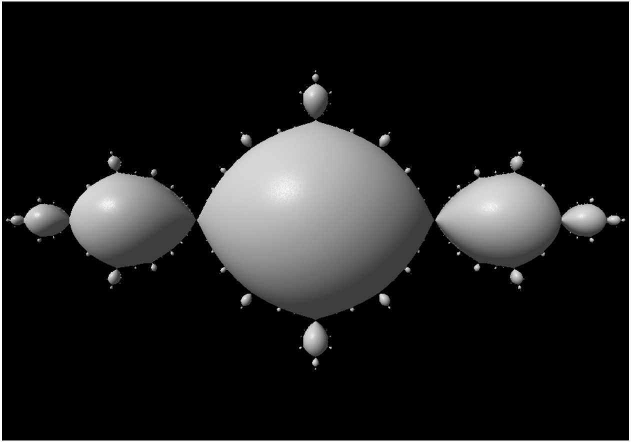
?



$\longleftrightarrow$

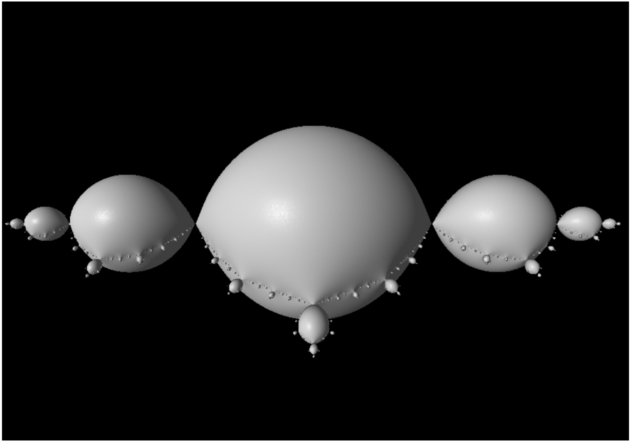
?

Inflated Julia sets



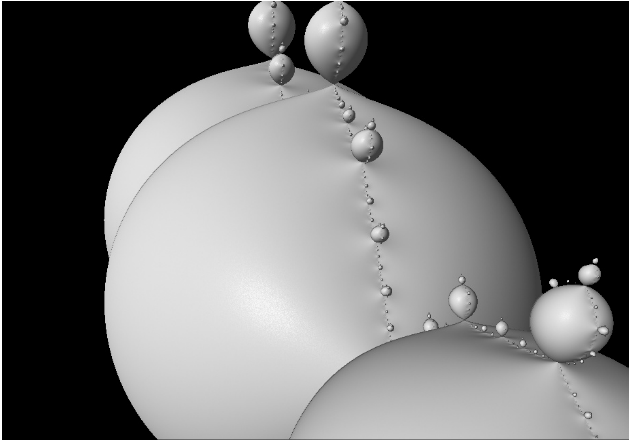
$$z_{n+1} = z_n^2 - 1$$

Inflated Julia sets



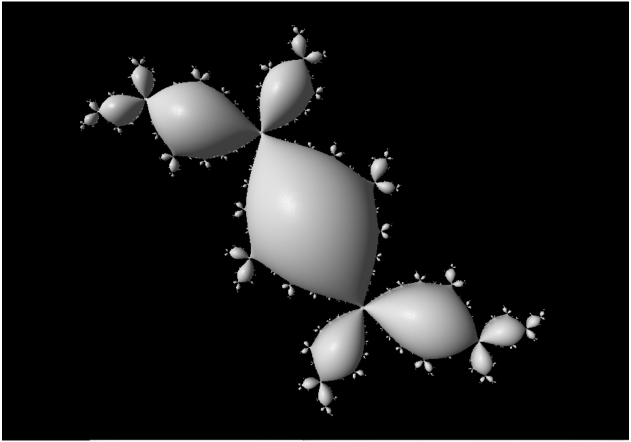
$$z_{n+1} = z_n^2 - 1$$

Inflated Julia sets



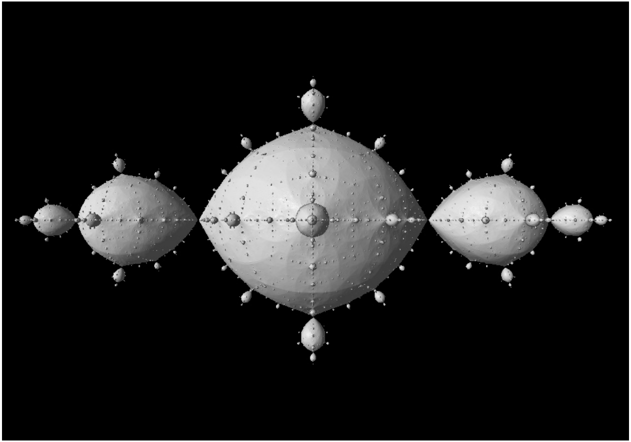
$$z_{n+1} = z_n^2 - 1$$

Inflated Julia sets



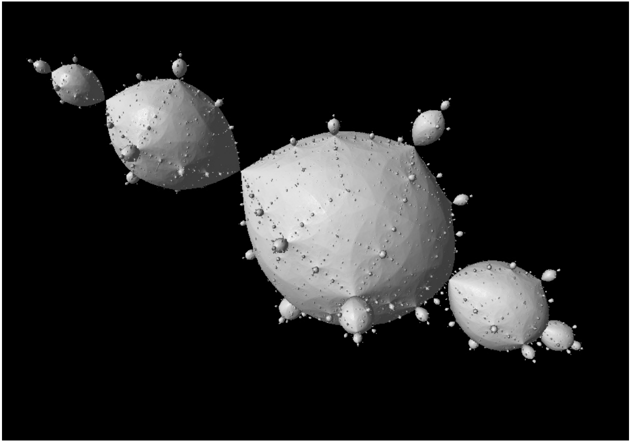
$$z_{n+1} = z_n^2 - 0.123 + 0.75\mathrm{i}$$

Preliminary 3D Julia sets



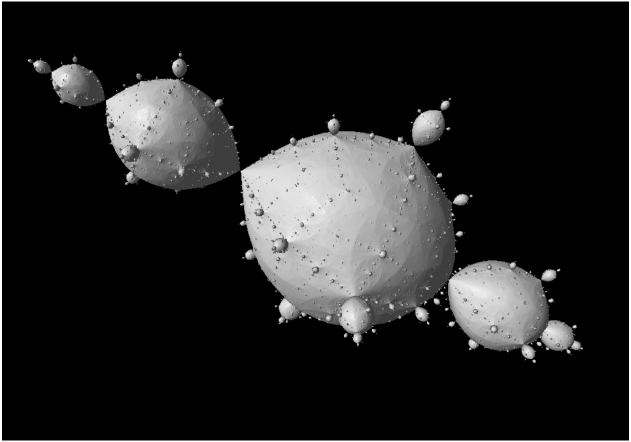
$$z_{n+1} = z_n^2 - 1$$

Preliminary 3D Julia sets



$$z_{n+1} = z_n^2 - 1$$

Preliminary 3D Julia sets



$$z_{n+1} = z_n^2 - 1$$

Thank you for your attention